

FRACTIONAL SOBOLEV INEQUALITIES: SYMMETRIZATION, ISOPERIMETRY AND INTERPOLATION

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ABSTRACT. We obtain new oscillation inequalities in metric spaces in terms of the Peetre K -functional and the isoperimetric profile. Applications provided include a detailed study of Fractional Sobolev inequalities and the Morrey-Sobolev embedding theorems in different contexts. In particular we include a detailed study of Gaussian measures as well as probability measures between Gaussian and exponential. We show a kind of reverse Pólya-Szegő principle that allows us to obtain continuity as a self improvement from boundedness, using symmetrization inequalities. Our methods also allow for precise estimates of growth envelopes of generalized Sobolev and Besov spaces on metric spaces. We also consider embeddings into BMO and their connection to Sobolev embeddings.

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1. Preface

This paper is devoted to the study of fractional Sobolev inequalities and Morrey-Sobolev type embedding theorems in metric spaces, using symmetrization. The connection with isoperimetry plays a crucial role. The aim was to provide a unified account and develop the theory in the general setting of metric measure spaces that satisfy suitable assumptions. In particular, the use of new pointwise inequalities connected with the isoperimetric profile allow us to treat the Euclidean and Gaussian and other measures in a unified way. The connection with Interpolation/Approximation theory also plays a crucial role and suggests further applications to optimization...

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CHAPTER 1

Introduction

In this paper we establish general versions of fractional Sobolev embeddings, including Morrey-Sobolev type embedding theorems, in the context of metric spaces, using symmetrization methods. The connection of the underlying inequalities with interpolation and isoperimetry plays a crucial role.

We shall consider connected, measure metric spaces (Ω, d, μ) equipped with a finite Borel measure μ . For measurable functions $u : \Omega \rightarrow \mathbb{R}$, the distribution function is defined by

$$\mu_u(t) = \mu\{x \in \Omega : u(x) > t\} \quad (t \in \mathbb{R}).$$

The signed **decreasing rearrangement** of u , which we denote by u_μ^* , is the right-continuous non-increasing function from $[0, \mu(\Omega))$ into \mathbb{R} that is equimeasurable with u ; i.e. u_μ^* satisfies

$$\mu_u(t) = \mu\{x \in \Omega : u(x) > t\} = m(\{s \in [0, \mu(\Omega)) : u_\mu^*(s) > t\}) \quad , \quad t \in \mathbb{R}$$

(where m denotes the Lebesgue measure on $[0, \mu(\Omega))$). The maximal average of u_μ^* is defined by

$$u_\mu^{**}(t) = \frac{1}{t} \int_0^t u_\mu^*(s) ds, \quad (t > 0).$$

For a Borel set $A \subset \Omega$, the **perimeter** or **Minkowski content** of A is defined by

$$P(A; \Omega) = \liminf_{h \rightarrow 0} \frac{\mu(\{x \in \Omega : d(x, A) < h\}) - \mu(A)}{h}.$$

The **isoperimetric profile** $I_\Omega(t), t \in (0, \mu(A))$, is maximal with respect to the inequality

$$(1.1) \quad I_\Omega(\mu(A)) \leq P(A; \Omega).$$

The starting point of the discussion are the rearrangement inequalities¹ of [69] and [70], where we showed that², under appropriate concavity assumptions on the isoperimetric profile I_Ω , for all Lipschitz function f on Ω (briefly $f \in Lip(\Omega)$),

$$(1.2) \quad |f|_\mu^{**}(t) - |f|_\mu^*(t) \leq \frac{t}{I(t)} |\nabla f|_\mu^{**}(t), \quad 0 < t < \mu(\Omega),$$

where

$$|\nabla f(x)| = \limsup_{d(x,y) \rightarrow 0} \frac{|f(x) - f(y)|}{d(x,y)}.$$

In fact, in [69] we showed that (1.2) is equivalent to (1.1).

Since the integrability properties do not change by rearrangements (i.e. integrability properties are *rearrangement invariant*), rearrangement inequalities are

¹For more detailed information we refer to Chapter 2 below.

²See also the extensive list of references provided in [69].

particularly useful to prove embedding of Sobolev spaces into rearrangement invariant spaces³. On the other hand, the use of rearrangement inequalities to study smoothness of functions is harder to implement. The main difficulty here is that while the classical Pólya-Szegő principle (cf. [60], [15]) roughly states that symmetrizations are smoothing, i.e. they preserve the (up to first order) smoothness of Sobolev/Besov functions, the converse does not hold in general. In other words, it is not immediate how to deduce smoothness properties of f from inequalities on f_μ^* . From this point of view, one could describe some of the methods we develop in this paper as “suitable converses to the Pólya-Szegő principle”.

As it turns out, related issues have been studied long ago, albeit in a less general context, by A. Garsia and his collaborators. The original impetus of Garsia’s group was to study the path continuity of certain stochastic processes (cf. [40], [44]); a classical topic in Probability theory. This task led Garsia et al. to obtain rearrangement inequalities for general moduli of continuity, including L^p or even Orlicz moduli of continuity. Moreover, in [43], [42], and elsewhere (cf. [41]), these symmetrization inequalities were also applied to problems in Harmonic Analysis and, in particular, to study the absolute convergence of Fourier series. From our point of view, a remarkable aspect of the approach of Garsia et al. (cf. [42]) is precisely that the sought continuity can be recovered using rearrangement inequalities. In other words, one can reinterpret this part of the Garsia-Rodemich analysis as an approach to the Morrey-Sobolev embedding theorem using rearrangement inequalities.

It will be instructive to show how Garsia’s analysis can be combined with (1.2). To fix ideas we consider the setting of Garsia-Rodemich: The metric measure space $((0, 1)^n, |\cdot|, dx)$ (that is $(0, 1)^n$ provided with the Euclidean distance and Lebesgue measure). For functions $f \in Lip(0, 1)^n$ inequality (1.2) takes the form⁴

$$|f|^{**}(t) - |f|^*(t) \leq c_n \frac{t}{\min(t, 1-t)^{1-1/n}} |\nabla f|^{**}(t), \quad 0 < t < 1.$$

In fact (see Chapter 5), the previous inequality remains true for all functions $f \in W_{L^p}^1(0, 1)^n$ (where $p > n$, and $W_{L^p}^1(0, 1)^n$ is the Sobolev space of real-valued weakly differentiable functions on $(0, 1)^n$ whose first-order derivatives belong to L^p). Moreover, as we shall see (cf. Chapter 4), the inequality also holds for (signed) rearrangements; i.e. for all $f \in W_{L^p}^1(0, 1)^n$, $p > n$, we have that,

$$(1.3) \quad f^{**}(t) - f^*(t) \leq c_n \frac{t}{\min(t, 1-t)^{1-1/n}} |\nabla f|^{**}(t), \quad 0 < t < 1.$$

³Roughly speaking, a rearrangement invariant space is a Banach function space where the norm of a function depends only on the μ -measure of its level sets.

⁴The rearrangement of f with respect to Lebesgue measure is simply denoted by f^* .

Integrating, and using the fundamental theorem of calculus⁵, we get

$$\begin{aligned}
f^{**}(0) - f^{**}(1) &= \int_0^1 (f^{**}(t) - f^*(t)) \frac{dt}{t} \\
&\leq c_n \int_0^1 |\nabla f|^{**}(t) \frac{dt}{\min(t, 1-t)^{1-1/n}} \\
&\leq c_{n,p} \|\nabla f\|_{L^p} \left\| \frac{1}{\min(t, 1-t)^{1-1/n}} \right\|_{L^{p'}} \quad (\text{by Hölder's inequality}) \\
&= C_{n,p} \|\nabla f\|_{L^p},
\end{aligned}$$

where the last inequality follows from the fact that for $p > n$, $\left\| \frac{1}{\min(t, 1-t)^{1-1/n}} \right\|_{L^{p'}} < \infty$. Summarizing our findings, we have

$$(1.4) \quad \operatorname{ess\,sup}_{x \in (0,1)^n} f - \int_0^1 f = f^{**}(0) - f^{**}(1) \leq C_{n,p} \|\nabla f\|_{L^p}.$$

Applying (1.4) to $-f$ yields

$$(1.5) \quad \int_0^1 f - \operatorname{ess\,inf}_{x \in (0,1)^n} f \leq C_{n,p} \|\nabla f\|_{L^p}.$$

Therefore, adding (1.4) and (1.5) we obtain

$$(1.6) \quad \operatorname{Osc}(f; (0,1)^n) := \operatorname{ess\,sup}_{x \in (0,1)^n} f - \operatorname{ess\,inf}_{x \in (0,1)^n} f \leq 2C_{n,p} \|\nabla f\|_{L^p}.$$

We have shown that (1.2) gives us good control of the oscillation of the original function on the whole cube $(0,1)^n$. To control the oscillation on any cube $Q \subset (0,1)^n$, we use a modification of an argument that originates in the work of Garsia et al. (cf. [43]). In the original one dimensional argument (cf. [43], [42]), one controls the oscillation of f in terms of an expression that involves the modulus of continuity, rather than the gradient. The idea is that if an inequality scales appropriately, one can re-scale. Namely, given two fixed points $x < y \in (0,1)$, one can apply the inequality at hand to the re-scaled function⁶ $\tilde{f}(t) = f(x + t(y-x))$, $t \in [0,1]$. This type of “change of scale argument” can be extended to the cube $(0,1)^n$, but for general domains becomes unmanageable. Therefore, we needed to reformulate the idea somewhat. From our point of view the idea is that if we control $|\nabla f|$ on $(0,1)^n$ then we ought to be able to control its restrictions. The issue then becomes: How do our inequalities scale under restrictions? Again for $(0,1)^n$ all goes well. In fact, if $f \in W_{L^p}^1((0,1)^n)$ then for any open cube $Q \subset (0,1)^n$ we have $f\chi_Q \in W_{L^p}^1(Q)$.

⁵Recall that $\frac{\partial}{\partial t}(f^{**}(t)) = -\frac{(f^{**}(t) - f^*(t))}{t}$.

⁶For example, consider the case $n = 1$. Given $0 < x < y < 1$, the inequality (1.6) applied to \tilde{f} yields

$$\begin{aligned}
\operatorname{ess\,sup}_{[x,y]} f - \operatorname{ess\,inf}_{[x,y]} f &\leq c_p |y-x| \left(\int_0^1 |f'(x + t(y-x))|^p dt \right)^{1/p} \\
&= c_p |y-x|^{1-1/p} \left(\int_0^1 |f'|^p \right)^{1/p}.
\end{aligned}$$

Moreover, the fundamental inequality (1.3) has the following scaling

$$(f\chi_Q)^{**}(t) - (f\chi_Q)^*(t) \leq c_n \frac{t}{\min(t, |Q| - t)^{1-1/n}} |\nabla(f\chi_Q)|^{**}(t), \quad 0 < t < |Q|.$$

Using the previous argument applied to $f\chi_Q$ we thus obtain

$$\text{Osc}(f; Q) \leq c_{n,p} \left\| \frac{t}{\min(t, |Q| - t)^{1-1/n}} \right\|_{L^{p'}(0, |Q|)} \|\nabla f\|_{L^p(Q)}.$$

By computation, and a classical argument, it is easy to see from here that (cf. Remark 5 in Chapter 5)

$$|f(y) - f(z)| \leq c_{n,p} |y - z|^{(1-\frac{n}{p})} \|\nabla f\|_p, \text{ a.e. } y, z.$$

When $p = n$ this argument fails, but nevertheless, by a simple modification, it yields a result due independently to Stein [88] and C. P. Calderón [23]: Namely, if $\|\nabla f\|_{L^{n,1}} < \infty$, then f is essentially continuous (cf. Remark 5, Chapter 5).

We will show that, with suitable technical adjustments, this method can be extended to the metric setting⁷. To understand the issues involved let us note that, since our inequalities are formulated in terms of isoperimetric profiles, to achieve the local control or “the change of scales (in our situation through the restrictions)” we need suitable control of the (relative) isoperimetric profiles on the (new) metric spaces obtained by restriction. More precisely, if $Q \subset \Omega$ is an open set, we shall consider the metric measure space $(Q, d|_Q, \mu|_Q)$. Then the problem we face is that, in general, the isoperimetric profile I_Q of $(Q, d|_Q, \mu|_Q)$ is different from the isoperimetric profile I_Ω of (Ω, d, μ) . What we needed is to control the “relative isoperimetric inequality”⁸ and make sure the corresponding inequalities scale appropriately. We say that an **isoperimetric inequality relative to G** holds, if there exists a positive constant C_G such that

$$I_G(s) \geq C_G \min(I_\Omega(s), I_\Omega(\mu(G) - s)).$$

We say that the metric measure space (Ω, d, μ) has the **relative isoperimetric property**, if for any $x \in \Omega$, there exists a positive number $\delta = \delta(x)$ and a constant C such that for any open ball $B_\alpha(x)$ centered on x , with $\mu(B_\alpha(x)) = \alpha$ ($0 < \alpha < \delta$), its **relative isoperimetric profile** $I_{B_\alpha(x)}$ satisfies:

$$I_{B_\alpha(x)}(s) \geq C \min(I_\Omega(s), I_\Omega(\alpha - s)), \quad 0 < s < \alpha.$$

For metric spaces (Ω, d, μ) satisfying the relative isoperimetric property we have the scaling that we need to apply the previous analysis. This theme is developed in detail in Chapter 4. The previous discussion implicitly shows that $(0, 1)^n$ has the relative isoperimetric property. We shall show in Chapter 5 that many familiar metric measure spaces also have the relative isoperimetric property.

⁷For a different approach to the Morrey-Sobolev theorem on metric spaces we refer to the work of Coulhon [28], [27].

⁸Recall that given an open set $G \subset \Omega$, and a set $A \subset G$, the **perimeter of A relative to G** (cf. Chapter 2) is defined by

$$P(A; G) = \liminf_{h \rightarrow 0} \frac{\mu(\{x \in G : d(x, A) < h\}) - \mu(A)}{h}.$$

The corresponding **relative isoperimetric profile** of $G \subset \Omega$ is given by

$$I_G(s) = I_{(G, d, \mu)}(s) = \inf \{P(A; G) : A \subset G, \mu(A) = s\}.$$

We now turn to the main objective of this paper which is to develop the corresponding theory for fractional order Besov-Sobolev spaces. This is, indeed, the original setting of Garsia's work, and our aim in this paper is to extend it to the metric setting. The first part of our program for Besov spaces was to formulate a suitable replacement of (1.2) for the fractional setting. To explain the peculiar form of the underlying inequalities that we need requires some preliminary background information.

Let $X = X(\mathbb{R}^n)$ be a rearrangement invariant space on \mathbb{R}^n , and let ω_X be the modulus of continuity associated with X defined for $g \in X$ by

$$\omega_X(t, g) = \sup_{|h| \leq t} \|g(\cdot + h) - g(\cdot)\|_X.$$

It is known (for increasing levels of generality see [43], [55], [64], [62] and the references therein), that there exists $c = c_n > 0$ such that, for all functions $f \in X(\mathbb{R}^n) + \dot{W}_X^1(\mathbb{R}^n)$,

$$(1.7) \quad |f|^{**}(t) - |f|^*(t) \leq c_n \frac{\omega_X(t^{1/n}, f)}{\phi_X(t)}, \quad t > 0,$$

where $\dot{W}_X^1(\mathbb{R}^n)$ is the homogeneous Sobolev space defined by means of the quasi norm $\|u\|_{\dot{W}_X^1(\mathbb{R}^n)} := \|\nabla u\|_{X(\mathbb{R}^n)}$, $\phi_X(t)$ is the fundamental function of X , and $|f|^*$ is the rearrangement of $|f|$ with respect to the Lebesgue measure⁹.

The inequalities we seek are extensions of (1.7) to the metric setting. Note that, in some sense, one can consider (1.7) as an extension, by interpolation, of (1.2). Therefore, it is natural to ask: How should (1.7) be reformulated in order to make sense for metric spaces? Not only we need a suitable substitute for the modulus of continuity ω_X , but a suitable re-interpretation of the factor " $t^{1/n}$ " is required as well. We now discuss these issues in detail.

There are several known alternative, although possibly non equivalent, definitions of modulus of continuity in the general setting of metric measure spaces (Ω, d, μ) (cf. [47] for the interpolation properties of Besov spaces on metric spaces). Given our background on approximation theory, it was natural for us to choose the universal object that is provided by interpolation/approximation theory, namely the Peetre K -functional. Indeed on \mathbb{R}^n , the Peetre K -functional is defined by:

$$K(t, f; X(\mathbb{R}^n), \dot{W}_X^1(\mathbb{R}^n)) := \inf\{\|f - g\|_X + t \|\nabla g\|_X : g \in \dot{W}_X^1(\mathbb{R}^n)\}.$$

Considering the K -functional is justified since it is well known that¹⁰ (cf. [13, Chapter 5, formula (4.41)])

$$K(t, f; X(\mathbb{R}^n), \dot{W}_X^1(\mathbb{R}^n)) \simeq \omega_X(t, f).$$

In the general case of metric measure spaces (Ω, d, μ) we shall consider:

$$K(t, f; X(\Omega), S_X(\Omega)) := \inf\{\|f - g\|_{X(\Omega)} + t \|\nabla g\|_{X(\Omega)} : g \in S_X(\Omega)\},$$

⁹In the background of inequalities of this type lies a form of the Pólya-Szegő principle that states that symmetric rearrangements do not increase Besov norms (cf. [3], [64] and the references therein).

¹⁰Here the symbol $f \simeq g$ indicates the existence of a universal constant $c > 0$ (independent of all parameters involved) such that $(1/c)f \leq g \leq cf$. Likewise the symbol $f \preceq g$ will mean that there exists a universal constant $c > 0$ (independent of all parameters involved) such that $f \leq cg$.

where $X(\Omega)$ is a r.i. space on Ω , and $S_X(\Omega) = \{f \in Lip(\Omega) : \|\nabla f\|_{X(\Omega)} < \infty\}$. We shall thus think of $K(t, f; X(\Omega), S_X(\Omega))$ as “a modulus of continuity”.

Now, given our experience with the inequality (1.2), we were led to conjecture the following reformulation¹¹ of (1.7): There exists a universal constant $c > 0$, such that for every r.i. space $X(\Omega)$, and for all $f \in X(\Omega) + S_X(\Omega)$, we have

$$(1.8) \quad |f|_{\mu}^{**}(t) - |f|_{\mu}^*(t) \leq c \frac{K\left(\frac{t}{I_{\Omega}(t)}, f; X(\Omega), S_X(\Omega)\right)}{\phi_X(t)}, \quad 0 < t < \mu(\Omega).$$

We presented this conjectural inequality when lecturing on the topic. In particular, we communicated the conjecture to M. Mastilo, who recently proved in [72] that indeed (1.8) holds for $t \in (0, 1/4)$, and for all rearrangement invariant spaces X that are “far away from L^1 and from L^∞ ”.

The result of [72], while in many respects satisfying, leaves some important questions open. Indeed, the restrictions placed on the range of t (i.e. the measure of the sets), as well as those placed on the spaces, precludes the investigation of the isoperimetric nature of (1.8). In particular, while the equivalence of (1.2) and the isoperimetric inequality (1.1) is known to hold (cf. [69]), the possible equivalence of (1.8) with the isoperimetric inequality (1.1) apparently cannot be answered without involving the space L^1 .

We will indeed show that (1.8) crucially holds for all values of t , and without restrictions on the function spaces. In particular, (1.8) holds in full generality for $X = L^1$. As a consequence we are able to prove the following fractional version of the celebrated Federer-Fleming-Maz'ya equivalence¹² (cf. [74])

THEOREM 1. *Let (Ω, d, μ) be a metric measure space such that (1.2) holds. Then,*

(i) *(cf. Theorem 7) There exists a constant $c > 0$ such that for all rearrangement invariant spaces $X(\Omega)$, and for all $f \in X(\Omega) + S_X(\Omega)$,*

$$(1.9) \quad |f|_{\mu}^{**}(t) - |f|_{\mu}^*(t) \leq 2 \frac{K\left(\frac{t}{I_{\Omega}(t)}, f; X(\Omega), S_X(\Omega)\right)}{\phi_X(t)}, \quad t \in (0, \mu(\Omega)).$$

(ii) *(cf. Theorem 11) In particular, for $X = L^1(\Omega)$, (1.9) is equivalent to the isoperimetric inequality*

$$I_{\Omega}(\mu(A)) \leq 2P(A, \Omega).$$

Using (1.9) as a starting point we can study embeddings of Besov spaces in metric spaces and, in particular, the corresponding Morrey-Sobolev-Besov embedding.

We now focus on the fractional Morrey-Sobolev theorem. We start by describing the inequalities of [42], [41], which for L^p spaces¹³ on $[0, 1]$ takes the following form

$$(1.10) \quad \left\{ \frac{f^*(x) - f^*(1/2)}{f^*(1/2) - f^*(1-x)} \right\} \leq c \int_x^1 \frac{\omega_{L^p}(t, f)}{t^{1/p}} \frac{dt}{t}, \quad x \in (0, \frac{1}{2}].$$

¹¹At least for the metric measure spaces (Ω, d, μ) considered in [69] (for which, in particular, (1.2) holds).

¹²Which claims the equivalence between the Gagliardo-Nirenberg inequality and the isoperimetric inequality.

¹³Importantly, Garsia-Rodemich also can deal with $X = L^p$, or $X = L_A$ (Orlicz space), our approach covers all r.i. spaces and works for a large class of metric spaces.

Letting $x \rightarrow 0$ in (1.10), adding the two inequalities then yields

$$\operatorname{ess\,sup}_{[0,1]} f - \operatorname{ess\,inf}_{[0,1]} f \leq c \int_0^1 \frac{\omega_{L^p}(t, f)}{t^{1/p}} \frac{dt}{t}.$$

Using the *change of scale argument* leads to

$$|f(x) - f(y)| \leq 2c \int_0^{|x-y|} \frac{\omega_{L^p}(t, f)}{t^{1/p}} \frac{dt}{t}; \quad x, y \in [0, 1],$$

from which the essential continuity¹⁴ of f is apparent if we know that $\int_0^1 \frac{\omega_X(t, f)}{\phi_X(t)} \frac{dt}{t} < \infty$. To obtain the n -dimensional version of (1.10) for $[0, 1]^n$, Garsia et al. had to develop deep combinatorial techniques. The corresponding n -dimensional inequality is given by (cf. [41, (3.6)] and the references therein)

$$\left. \begin{aligned} f^*(x) - f^*(1/2) \\ f^*(1/2) - f^*(1-x) \end{aligned} \right\} \leq c \int_{x^{1/n}}^1 \frac{\omega_{L^p}(t, f)}{t^{1/p}} \frac{dt}{t}, \quad x \in (0, \frac{1}{2}]$$

which by the now familiar argument yields

$$(1.11) \quad |f(x) - f(y)| \leq C_{p,n} \int_0^{|x-y|} \frac{\omega_{L^p}(t, f)}{t^{n/p}} \frac{dt}{t}; \quad x, y \in [0, \frac{1}{2}]^n.$$

However, as pointed out above, the change of scale technique is apparently not available for more general domains. Moreover, as witnessed by the difficulties already encountered by Garsia et al. to prove inequalities on n -dimensional cubes, it was not even clear at that time what form the rearrangement inequalities would take in general. In particular, Garsia et al. do not use isoperimetry.

For more general function spaces we need to reformulate Theorem 1 above as follows (cf. Chapter 3)

THEOREM 2. (cf. Chapter 3, Theorem 10) *Let X be a r.i. space on Ω . Then, for each $f \in X + S_X(\Omega)$*

$$|f|_\mu^{**}(t) - |f|_\mu^*(t) \leq \frac{K(\psi(t), f; X, S_X(\Omega))}{\phi_X(t)}, \quad 0 < t < \mu(\Omega),$$

where

$$\psi(t) = \frac{\phi_X(t)}{t} \left\| \frac{s}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'},$$

(here \bar{X}' denotes the associated space [see Chapter 2]).

REMARK 1. Note that if $X = L^1$ we recover, as we should, (1.9).

A second step of our program is the routine, but crucially important, reformulation of rearrangement inequalities using signed rearrangements. Once this is done, our generalized Morrey-Sobolev-Garsia-Rodemich theorem can be stated as follows

THEOREM 3. (cf. Chapter 4, Theorem 13) *Let (Ω, d, μ) be a probability metric space which has the relative isoperimetric property. Let X be a r.i. space in Ω such that*

$$\left\| \frac{1}{I_\Omega(s)} \right\|_{\bar{X}'} < \infty.$$

¹⁴An application of Hölder's inequality also yields Lip conditions.

If $f \in X + S_X(\Omega)$ satisfies

$$\int_0^{\mu(\Omega)} \frac{K\left(\phi_X(t) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f; X, S_X(\Omega)\right) dt}{\phi_X(t)} \frac{dt}{t} < \infty,$$

then f is essentially bounded and essentially continuous.

To see the connection of our inequalities with those of Garsia et al. let us note that integrating the corresponding version of (1.9) for signed rearrangements from $t_0 < t_1$, combined with the fact that $\frac{d}{dt}(-f_\mu^{**}(t)) = \frac{f_\mu^{**}(t) - f_\mu^*(t)}{t}$, and the fundamental theorem of Calculus, yields

$$(1.12) \quad f_\mu^{**}(t_1) - f_\mu^{**}(t_2) \leq c \int_{t_1}^{t_2} \frac{K\left(\frac{t}{I(t)}, f; X(\Omega), S_X(\Omega)\right) dt}{\phi_X(t)} \frac{dt}{t}, \quad 0 < t_1 < t_2 < 1.$$

Letting $t_1 \rightarrow 0, t_2 \rightarrow 1$, it follows that

$$(1.13) \quad \operatorname{ess\,sup}_\Omega f - \int_\Omega f \leq c \int_0^1 \frac{K\left(\frac{t}{I(t)}, f; X(\Omega), S_X(\Omega)\right) dt}{\phi_X(t)} \frac{dt}{t}.$$

Applying (1.13) to $-f$, and adding, we obtain

$$\operatorname{ess\,sup}_\Omega f - \operatorname{ess\,inf}_\Omega f \leq c \int_0^1 \frac{K\left(\frac{t}{I(t)}, f; X(\Omega), S_X(\Omega)\right) dt}{\phi_X(t)} \frac{dt}{t}.$$

From this point we can proceed to study the continuity or Lip properties of f using the arguments¹⁵ outlined above.

Next let us consider the difference between the right hand sides of (1.9) and (1.12)). From the point of view of embeddings the difference between these inequalities is connected with limiting cases of the Sobolev-Besov embeddings and the role of BMO . Note that the inequality (1.11) can be reformulated as the embedding of $B_{p,1}^{n/p}([0,1]^n)$ into the space of continuous functions $C([0,1]^n)$

$$(1.14) \quad B_{p,1}^{n/p}([0,1]^n) \subset C([0,1]^n), \text{ where } n/p < 1.$$

Moreover, since

$$\omega_{L^p}(t, f) \leq c_{p,n} t \|f\|_{W_{L^p}^1},$$

we also have

$$W_{L^p}^1([0,1]^n) \subset B_{p,1}^{n/p}([0,1]^n).$$

Therefore, (1.14) implies the (Morrey-Sobolev) continuity of Sobolev functions in W_p^1 when $p > n$. On the other hand, if we consider the Besov condition defined by

¹⁵Interestingly the one dimensional case studied by Garsia et al somehow does not follow directly since the isoperimetric profile for the unit interval $(0,1)$ is 1. Therefore in this case the isoperimetric profile does not satisfy the assumptions of [69]. Nevertheless, our inequalities remain true and provide an alternate approach to the Garsia inequalities. See Chapter 10 below for complete details.

the right hand side of (1.7), when $X = L^p$, and $n/p < 1$, we find¹⁶

$$\|f\|_{B_{p,\infty}^{n/p}([0,1]^n)} = \sup_{t \in [0,1]} \frac{\omega_{L^p}(t, f)}{t^{n/p}}.$$

Now for functions in $B_{p,\infty}^{n/p}([0,1]^n)$ we don't expect boundedness, and in fact, apparently the best we can say directly from our rearrangement inequalities, follows from (1.7):

$$(1.15) \quad \sup_{[0,1]} (f^{**}(t) - f^*(t)) \leq c \|f\|_{B_{p,\infty}^{n/p}([0,1]^n)}.$$

In view of the celebrated result of Bennett-DeVore-Sharpely [11] (cf. also [13]) that characterizes the rearrangement invariant hull of BMO , via the left hand side of (1.15), we see that (1.15) gives $f \in B_{p,\infty}^{n/p}([0,1]^n) \Rightarrow f^* \in BMO[0,1]$.

Again, the direct argument with the rearrangement inequality gives a weaker result than the well known, and easy to prove embedding¹⁷,

$$B_{p,\infty}^{n/p}([0,1]^n) \subset BMO$$

or more precisely,

$$B_{p,\infty}^{n/p}([0,1]^n) \subset BLO.$$

Therefore, it becomes irresistible to ask if the change of scale/relative isoperimetry argument can be extended to the setting of BMO ! Indeed, this is the case and in Chapter 7, Theorem 25 we show the following

THEOREM 4. *Let Q be a cube with sides parallel to the coordinate axes in \mathbb{R}^n . Then, for all open subcubes $Q' \subset Q$*

(i)

$$\sup_{Q' \subset Q} ((f\chi_{Q'})^{**}(|Q'|) - (f\chi_{Q'})^*(|Q'|)) \leq c \|f\|_{B_{p,\infty}^{n/p}(Q)}.$$

(ii) *(converse to the Bennett-DeVore-Sharpely [11] inequality)*

$$\|f\|_{BMO} \leq c \sup_{Q' \subset Q} ((f\chi_{Q'})^{**}(|Q'|) - (f\chi_{Q'})^*(|Q'|)).$$

Like in the work of Garsia et al., the signed rearrangement plays a crucial role in the study of oscillations and Lip conditions. This is not a coincidence; for recall the well known interpretation of BMO as a limiting *Lip* condition. This can be seen by means of writing Lip_α conditions on a fixed Euclidean cube Q as

$$\|f\|_{Lip_\alpha} \simeq \sup_{Q' \subset Q} \frac{1}{|Q'|^{1-\alpha/n}} \int_{Q'} |f - f_{Q'}| dx < \infty.$$

In this fashion BMO appears as the limiting case of Lip_α conditions when $\alpha \rightarrow 0$. With this intuition at hand we were led to formulate the corresponding version of Theorem 1 for BMO . It reads as follows¹⁸

¹⁶Note that we have

$$\|f\|_{B_{p,\infty}^{n/p}([0,1]^n)} \leq c_{p,n} \|f\|_{B_{p,1}^{n/p}([0,1]^n)}.$$

¹⁷This follows from Theorem 4 below.

¹⁸In particular, we arrive, albeit through a very different route than the original, to an extension of an inequality of Bennett-DeVore-Sharpely (cf. [13, combine Theorem 7.3 and Theorem 8.8]) for the space L^1 .

THEOREM 5. (*cf. Chapter 7, Theorem 27*) Suppose that (Ω, d, μ) is a metric measure space such that the Bennett-DeVore-Sharpley inequality

$$(1.16) \quad \sup_t (f_\mu^{**}(t) - f_\mu^*(t)) \leq c \|f\|_*,$$

holds, where $\|f\|_*$ is a suitable BMO “norm”. Then,

$$f_\mu^{**}(t) - f_\mu^*(t) \leq c \frac{K(\phi_X(t), f; X(\Omega), BMO(\Omega))}{\phi_X(t)}, \quad 0 < t < \mu(\Omega).$$

From this point of view the Bennett-DeVore-Sharpley inequality (1.16) takes the role of our basic inequality (1.2). In this respect it is important to note that (1.16) has been shown to hold in great generality, for example it holds for doubling measures (cf. [84]). Finally, in connection with BMO, we considered the role of signed rearrangements. Here the import of this notion is that signed rearrangements provide a theoretical method to compute medians (cf. Theorem 26) and thus quickly lead to a version of the limiting case of the John-Stromberg-Jawerth-Torchinsky inequality (cf. Chapter 7, (2.1)).

Our approach to the (Morrey-)Sobolev embedding theorem also leads to the consideration of “Lorentz spaces with negative indices” (cf. Chapter 9), providing still a very suggestive approach¹⁹ to these results, at least in the Euclidean case.

In Chapter 5 and Chapter 6 we have considered explicit versions of our results in different classical contexts. In particular, in Chapter 6, we obtain new fractional Sobolev inequalities for Gaussian measures, as well as for probability measures that are in between Gaussian and exponential. For example, for Gaussian measure on \mathbb{R}^n , we have for $1 \leq q < \infty, \theta \in (0, 1)$

$$\left\{ \int_0^{1/2} |f|_{\gamma_n}^*(t)^q \left(\log \frac{1}{t} \right)^{\frac{q\theta}{2}} dt \right\}^{1/q} \leq c \|f\|_{B_{L^q}^{\theta, q}(\gamma_n)},$$

where c is independent of the dimension. Likewise, the same proof yields that for probability measures on the real line of the form²⁰

$$d\mu_r(x) = \exp(-|x|^r) dx, \quad r \in (1, 2]$$

and their tensor products

$$\mu_{r,n} = \mu_p^{\otimes n},$$

we have

$$\left\{ \int_0^{1/2} |f|_{\mu_{r,n}}^*(t)^q \left(\log \frac{1}{t} \right)^{q\theta(1-1/r)} dt \right\}^{1/q} \leq c \|f\|_{B_{L^q}^{\theta, q}(\mu_{r,n})}.$$

We refer to Chapter 6 for the details, where the reader will also find a treatment of the case $q = \infty$, which yields the corresponding improvements on the exponential integrability:

$$\sup_{t \in (0, \frac{1}{2})} \left(|f|_{\mu_{r,n}}^{**}(t) - |f|_{\mu_{r,n}}^*(t) \right) \left(\log \frac{1}{t} \right)^{(1-\frac{1}{r})\theta} \leq c \|f\|_{\dot{B}_{L^\infty}^{\theta, \infty}(\mu_{r,n})}.$$

¹⁹Although in paper we only concentrate on the role that these spaces play on the theory of embeddings, one cannot but feel that a detailed study of these spaces could be useful for other questions connected with interpolation/approximation.

²⁰where Z_r^{-1} is a normalizing constant.

Applications to the computation of envelopes of function spaces in the sense of Triebel-Haroske and their school are provided in Chapter 8.

The table of contents will serve to show the organization of the paper. We have tried to make the reading of the chapters in the second part of the paper as independent of each other as possible.

CHAPTER 2

Preliminaries

1. Background

Our notation in the paper will be for the most part standard. We shall consider connected measure metric spaces (Ω, d, μ) equipped with a finite Borel measure μ , which we shall simply refer to, as “measure metric spaces”. For measurable functions $u : \Omega \rightarrow \mathbb{R}$, the distribution function of u is given by

$$\mu_u(t) = \mu\{x \in \Omega : u(x) > t\} \quad (t \in \mathbb{R}).$$

The signed **decreasing rearrangement**¹ of a function u is the right-continuous non-increasing function from $[0, \mu(\Omega))$ into \mathbb{R} which is equimeasurable with u . It can be defined by the formula

$$u_\mu^*(s) = \inf\{t \geq 0 : \mu_u(t) \leq s\}, \quad s \in [0, \mu(\Omega)),$$

and satisfies

$$\mu_u(t) = \mu\{x \in \Omega : u(x) > t\} = m\{s \in [0, \mu(\Omega)) : u_\mu^*(s) > t\}, \quad t \in \mathbb{R}$$

(where m denotes the Lebesgue measure on $[0, \mu(\Omega))$). It follows from the definition that

$$(u + v)_\mu^*(s) \leq u_\mu^*(s/2) + v_\mu^*(s/2).$$

Moreover,

$$u_\mu^*(0^+) = \operatorname{ess\,sup}_\Omega u \quad \text{and} \quad u_\mu^*(\mu(\Omega)^-) = \operatorname{ess\,inf}_\Omega u.$$

The maximal average $u_\mu^{**}(t)$ is defined by

$$u_\mu^{**}(t) = \frac{1}{t} \int_0^t u_\mu^*(s) ds = \frac{1}{t} \sup \left\{ \int_E u(s) d\mu : \mu(E) = t \right\}, \quad t > 0.$$

The operation $u \rightarrow u_\mu^{**}$ is sub-additive, i.e.

$$(1.1) \quad (u + v)_\mu^{**}(s) \leq u_\mu^{**}(s) + v_\mu^{**}(s).$$

Moreover, since u_μ^* is decreasing, u_μ^{**} is also decreasing and $u_\mu^* \leq u_\mu^{**}$.

The following lemma proved in [43, Lemma 2.1] will be useful in what follows.

LEMMA 1. *Let f and f_n , $n = 1, \dots$, be integrable on Ω . Suppose that*

$$\lim_n \int_\Omega |f_n(x) - f(x)| d\mu = 0.$$

¹Note that this notation is somewhat unconventional. In the literature it is common to denote the decreasing rearrangement of $|u|$ by u_μ^* , while here it is denoted by $|u_\mu|^*$ since we need to distinguish between the rearrangements of u and $|u|$. In particular, the rearrangement of u can be negative. We refer the reader to [90] and the references quoted therein for a complete treatment.

Then

$$(f_n)_\mu^{**}(t) \rightarrow f_\mu^{**}(t), \text{ uniformly for } t \in [0, \mu(\Omega)], \text{ and} \\ (f_n)_\mu^*(t) \rightarrow f_\mu^*(t) \text{ at all points of continuity of } f_\mu^*.$$

When the measure is clear from the context, or when we are dealing with Lebesgue measure, we may simply write u^* and u^{**} , etc.

For a Borel set $A \subset \Omega$, the **perimeter** or **Minkowski content** of A is defined by

$$P(A; \Omega) = \liminf_{h \rightarrow 0} \frac{\mu(A_h) - \mu(A)}{h},$$

where $A_h = \{x \in \Omega : d(x, A) < h\}$ is the open h -neighborhood of A .

The **isoperimetric profile** is defined by

$$I_\Omega(s) = I_{(\Omega, d, \mu)}(s) = \inf \{P(A; \Omega) : \mu(A) = s\},$$

i.e. $I_{(\Omega, d, \mu)} : [0, \mu(\Omega)] \rightarrow [0, \infty)$ is the pointwise maximal function such that

$$(1.2) \quad P(A; \Omega) \geq I_\Omega(\mu(A)),$$

holds for all Borel sets A . A set A for which equality in (1.2) is attained will be called an **isoperimetric domain**. Again when no confusion arises we shall drop the subindex Ω and simply write I .

Unless specific mention to the contrary we will always assume that the metric measure spaces (Ω, d, μ) considered satisfy the following condition

CONDITION 1. *We will assume throughout the paper that our metric measure spaces (Ω, d, μ) are such that the isoperimetric profile $I_{(\Omega, d, \mu)}$ is a concave continuous function, increasing on $(0, \mu(\Omega)/2)$, symmetric about the point $\mu(\Omega)/2$ that, moreover, vanishes at zero. We remark that these assumptions are fulfilled for a large class of metric measure spaces².*

A continuous, concave function, $J : [0, \mu(\Omega)] \rightarrow [0, \infty)$, increasing on $(0, \mu(\Omega)/2)$ and symmetric about the point $\mu(\Omega)/2$, with $I(0) = 0$, and such that

$$I_\Omega \geq J,$$

will be called an **isoperimetric estimator** for (Ω, d, μ) .

For a Lipschitz function f on Ω (briefly $f \in Lip(\Omega)$) we define the **modulus of the gradient** by³

$$|\nabla f(x)| = \limsup_{d(x, y) \rightarrow 0} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Let us recall some results that relate isoperimetry and rearrangements (see [69], [64]).

THEOREM 6. *The following statements are equivalent*

- (1) *Isoperimetric inequality: $\forall A \subset \Omega$, Borel set,*

$$P(A; \Omega) \geq I_\Omega(\mu(A)).$$

²These assumptions are satisfied for the classical examples (cf. [18], [76], [10] and the references therein)

³In fact one can define $|\nabla f|$ for functions f that are Lipschitz on every ball in (Ω, d) (cf. [18, pp. 2, 3] for more details).

(2) *Oscillation inequality*: $\forall f \in Lip(\Omega)$,

$$(1.3) \quad (|f|_\mu^{**}(t) - |f|_\mu^*(t)) \frac{I_\Omega(t)}{t} \leq \frac{1}{t} \int_0^t |\nabla f|_\mu^*(s) ds, \quad 0 < t < \mu(\Omega).$$

LEMMA 2. *Let h be a bounded Lip function on Ω . Then there exists a sequence of bounded functions $(h_n)_n \subset Lip(\Omega)$, such that*

(1)

$$(1.4) \quad |\nabla h_n(x)| \leq (1 + \frac{1}{n}) |\nabla h(x)|, \quad x \in \Omega.$$

(2)

$$(1.5) \quad h_n \xrightarrow[n \rightarrow 0]{} h \text{ in } L^1.$$

(3)

$$(1.6) \quad \int_0^t \left| \left(-|h_n|_\mu^* \right)'(\cdot) I_\Omega(\cdot) \right|^* (s) ds \leq \int_0^t |\nabla h_n|_\mu^*(s) ds, \quad 0 < t < \mu(\Omega).$$

(The second rearrangement on the left hand side is with respect to the Lebesgue measure on $[0, \mu(\Omega))$).

2. Rearrangement invariant spaces

We recall briefly the basic definitions and conventions we use from the theory of rearrangement-invariant (r.i.) spaces, and refer the reader to [13] and [58] for a complete treatment.

Let $X = X(\Omega)$ be a Banach function space on (Ω, d, μ) , with the Fatou property⁴. We shall say that X is a **rearrangement-invariant** (r.i.) space, if $g \in X$ implies that all μ -measurable functions f with $|f|_\mu^* = |g|_\mu^*$, also belong to X and moreover, $\|f\|_X = \|g\|_X$. The functional $\|\cdot\|_X$ will be called a rearrangement invariant norm. Typical examples of r.i. spaces are the L^p -spaces, Orlicz spaces, Lorentz spaces, Marcinkiewicz spaces, etc.

On account of the fact that $\mu(\Omega) < \infty$, for any r.i. space $X(\Omega)$ we have

$$L^\infty(\Omega) \subset X(\Omega) \subset L^1(\Omega),$$

with continuous embeddings.

A useful property of r.i. spaces states that if

$$\int_0^t |f|_\mu^*(s) ds \leq \int_0^t |g|_\mu^*(s) ds,$$

holds for all $t > 0$, then, for any r.i. space X ,

$$\|f\|_X \leq \|g\|_X.$$

The associated space $X'(\Omega)$ is defined using the r.i. norm given by

$$\|h\|_{X'(\Omega)} = \sup_{g \neq 0} \frac{\int_\Omega |g(x)h(x)| d\mu}{\|g\|_{X(\Omega)}} = \sup_{g \neq 0} \frac{\int_0^{\mu(\Omega)} |h|_\mu^*(s) |g|_\mu^*(s) ds}{\|g\|_{X(\Omega)}}.$$

In particular, the following **generalized Hölder's inequality** holds

⁴This means that if $f_n \geq 0$, and $f_n \uparrow f$, then $\|f_n\|_X \uparrow \|f\|_X$ (i.e. Fatou's Lemma holds in the X norm).

$$(2.1) \quad \int_{\Omega} |g(x)h(x)| d\mu \leq \|g\|_{X(\Omega)} \|h\|_{X'(\Omega)}.$$

The **fundamental function** of $X(\Omega)$ is defined by

$$\phi_X(s) = \|\chi_E\|_X, \quad 0 \leq s \leq \mu(\Omega),$$

where E is any measurable subset of Ω with $\mu(E) = s$. We can assume without loss of generality that ϕ_X is concave (cf. [13]). Moreover,

$$(2.2) \quad \phi_{X'}(s)\phi_X(s) = s.$$

For example, if $X = L^p$ or $X = L^{p,q}$ (a Lorentz space), then $\phi_{L^p}(t) = \phi_{L^{p,q}}(t) = t^{1/p}$, if $1 \leq p < \infty$, while for $p = \infty$, $\phi_{L^\infty}(t) \equiv 1$. If N is a Young's function, then the fundamental function of the Orlicz space $X = L_N$ is given by $\phi_{L_N}(t) = 1/N^{-1}(1/t)$.

The Lorentz $\Lambda(X)$ space and the Marcinkiewicz space $M(X)$ associated with X are defined by the quasi-norms

$$(2.3) \quad \|f\|_{M(X)} = \sup_t f_\mu^*(t)\phi_X(t), \quad \|f\|_{\Lambda(X)} = \int_0^{\mu(\Omega)} f_\mu^*(t)d\phi_X(t).$$

Notice that

$$\phi_{M(X)}(t) = \phi_{\Lambda(X)}(t) = \phi_X(t),$$

and, moreover,

$$(2.4) \quad \Lambda(X) \subset X \subset M(X).$$

Let $X(\Omega)$ be a r.i. space, then there exists a **unique** r.i. space $\bar{X} = \bar{X}(0, \mu(\Omega))$ on $((0, \mu(\Omega)), m)$, (m denotes the Lebesgue measure on the interval $(0, \mu(\Omega))$) such that

$$(2.5) \quad \|f\|_{X(\Omega)} = \| |f|_\mu^* \|_{\bar{X}(0, \mu(\Omega))}.$$

\bar{X} is called the **representation space** of $X(\Omega)$. The explicit norm of $\bar{X}(0, \mu(\Omega))$ is given by (see [13, Theorem 4.10 and subsequent remarks])

$$\|h\|_{\bar{X}(0, \mu(\Omega))} = \sup \left\{ \int_0^{\mu(\Omega)} |h|^*(s) |g|_\mu^*(s) ds : \|g\|_{X'(\Omega)} \leq 1 \right\}$$

(the first rearrangement is with respect to the Lebesgue measure on $[0, \mu(\Omega))$).

Classically conditions on r.i. spaces can be formulated in terms of the boundedness of the Hardy operators defined by

$$Pf(t) = \frac{1}{t} \int_0^t f(s)ds; \quad Qf(t) = \int_t^{\mu(\Omega)} f(s) \frac{ds}{s}.$$

The boundedness of these operators on r.i. spaces can be best described in terms of the so called **Boyd indices**⁵ defined by

$$\bar{\alpha}_X = \inf_{s>1} \frac{\ln h_X(s)}{\ln s} \quad \text{and} \quad \underline{\alpha}_X = \sup_{s<1} \frac{\ln h_X(s)}{\ln s},$$

where $h_X(s)$ denotes the norm of the compression/dilation operator E_s on \bar{X} , defined for $s > 0$, by

$$E_s f(t) = \begin{cases} f^*(\frac{t}{s}) & 0 < t < s, \\ 0 & s < t < \mu(\Omega) \end{cases}.$$

⁵Introduced by D.W. Boyd in [21].

The operator E_s is bounded on \bar{X} on every r.i. space $X(\Omega)$, and moreover,

$$(2.6) \quad h_X(s) \leq \max(1, s), \text{ for all } s > 0.$$

For example, if $X = L^p$, then $\bar{\alpha}_{L^p} = \underline{\alpha}_{L^p} = \frac{1}{p}$. It is well known that

$$(2.7) \quad \begin{aligned} P \text{ is bounded on } \bar{X} &\Leftrightarrow \bar{\alpha}_X < 1, \\ Q \text{ is bounded on } \bar{X} &\Leftrightarrow \underline{\alpha}_X > 0. \end{aligned}$$

We shall also need to consider the restriction of functions of the r.i. space $X(\Omega)$ to measurable subsets $G \subset \Omega$ with $\mu(G) \neq 0$. To this end we shall consider G as metric measure space $(G, d|_G, \mu|_G)$ where the distance and the measure are obtained by the restrictions of the distance d and the measure μ to G . Given a $\mu|_G$ measurable function $u : G \rightarrow \mathbb{R}$, we let $\tilde{u} : \Omega \rightarrow \mathbb{R}$, be its extension to Ω defined by

$$\tilde{u}(x) = \begin{cases} u(x) & x \in G, \\ 0 & x \in \Omega \setminus G. \end{cases}$$

Obviously \tilde{u} is a μ -measurable function. We define

$$X_r(G) = \{u : G \rightarrow \mathbb{R}, \mu|_G \text{ measurable, such that } \|u\|_{X_r(G)} < \infty\},$$

where

$$\|u\|_{X_r(G)} := \|u\|_{X(\Omega)}.$$

It is plain all the properties of the definition of a r.i. space are satisfied, i.e. $\|\cdot\|_{X_r(G)}$ defines a rearrangement invariant norm on $(G, d|_G, \mu|_G)$.

We collect some elementary properties of $X_r(G)$.

PROPOSITION 1. *Let $X(\Omega)$ be a r.i. space on Ω , and let G be a measurable subset of Ω with $\mu(G) \neq 0$. Then,*

- (1) *If $u \in X(\Omega)$, then $u\chi_G \in X_r(G)$ and*

$$\|u\chi_G\|_{X_r(G)} \leq \|u\|_{X(\Omega)}.$$

- (2) *Let \bar{X}_r be the representation space of $X_r(G)$ and let \bar{X} be the representation space of $X(\Omega)$. Let $u \in X_r(G)$. Then*

$$\|u\|_{X_r(G)} = \left\| \widetilde{(|u|_{\mu|_G}^*)} \right\|_{\bar{X}},$$

where given $h : (0, \mu(G)) \rightarrow (0, \infty)$, \tilde{h} denotes its continuation by 0 outside $(0, \mu(G))$. Thus by the uniqueness of the representation space, if $h \in \bar{X}_r$, then

$$\|h\|_{\bar{X}_r} = \left\| \tilde{h} \right\|_{\bar{X}}.$$

- (3) *The fundamental function of $X_r(G)$ is given by*

$$(2.8) \quad \phi_{X_r(G)}(s) = \phi_{X(\Omega)}(s) \quad (0 \leq s \leq \mu(G)).$$

- (4) *Let $(X_r(G))'$ be the associated space of $X_r(G)$. Then*

$$(2.9) \quad (X_r(G))' = (X(\Omega))'_r(G).$$

PROOF. Part 1 is elementary. For 2, if $u \in X_r(G)$, then

$$\begin{aligned} \|u\|_{X_r(G)} &= \|\tilde{u}\|_{X(\Omega)} \\ &= \left\| (|\tilde{u}|_{\mu})^* \right\|_{\bar{X}} \quad (\text{by (2.5)}). \end{aligned}$$

Since $\mu|_G = \mu$ on G , it follows from the definition of \tilde{u} that

$$(|\tilde{u}|)_\mu^* = \begin{cases} (|u|)_{\mu|_G}^*(s) & s \in (0, \mu(G)), \\ 0 & s \in (\mu(G), \mu(\Omega)). \end{cases}$$

Thus

$$\left\| (|\tilde{u}|)_\mu^* \right\|_{\bar{X}} = \left\| (|u|)_{\mu|_G}^* \chi_{(0, \mu(G))} \right\|_{\bar{X}} = \left\| \widetilde{(|u|_{\mu|_G}^*)} \right\|_{\bar{X}}.$$

Part 3 follows from part 2 taking in account that

$$\phi_{X(\Omega)}(s) = \phi_{\bar{X}}(s).$$

Part 4, by definition,

$$\|u\|_{(X(\Omega)')_r(G)} = \|\tilde{u}\|_{X'(\Omega)}.$$

On the other hand,

$$\begin{aligned} \|\tilde{u}\|_{X'(\Omega)} &= \sup_{g \neq 0} \frac{\int_\Omega |g(x)\tilde{u}(x)| d\mu}{\|g\|_{X(\Omega)}} \\ &= \sup_{\substack{g \neq 0 \\ g=0 \text{ in } \Omega \setminus G}} \frac{\int_\Omega |g(x)\tilde{u}(x)| d\mu}{\|g\|_{X(\Omega)}} \\ &= \sup_{g \neq 0} \frac{\int_\Omega |\tilde{g}(x)\tilde{u}(x)| d\mu}{\|\tilde{g}\|_{X(\Omega)}} \\ &= \sup_{g \neq 0} \frac{\int_G |g(x)u(x)| d\mu}{\|g\|_{X(G)}} \\ &= \|u\|_{(X_r(G))'}. \end{aligned}$$

□

REMARK 2. An equivalent norm on $X_r(G)$ is given by the quotient norm

$$\|u\|_{*, X_r(G)} = \inf \{ \|U\|_{X(\Omega)} : U \in X(\Omega), \text{ and } U\chi_G = u \}.$$

In what follows, when G is clear from the context, we use the notation \tilde{u} to denote the extension by zero of a function originally defined on G .

3. Some remarks about Sobolev spaces

Let (Ω, d, μ) be a connected metric measure space with finite measure. Let X be a r.i. space on Ω . Let $S_X = S_X(\Omega) = \{f \in Lip(\Omega) : |\nabla f| \in X(\Omega)\}$, equipped with the seminorm

$$\|f\|_{S_X} = \|\nabla f\|_X.$$

At some point in our development we also need to consider restrictions of Sobolev functions. Let $G \subset \Omega$, is an open subset, we define $S_{X_r}(G) := \{u : G \rightarrow \mathbb{R}, \text{ such that } \exists f \in S_X(\Omega) \text{ such that } u = f\chi_G\}$, with

$$\|u\|_{S_{X_r}(G)} = \|\nabla u\|_{X_r(G)} = \left\| \widetilde{|\nabla u|} \right\|_{X(G)}.$$

In particular, if $f \in S_X(\Omega)$, then $f\chi_G \in S_{X_r}(G)$, and

$$\|f\chi_G\|_{S_{X_r}(G)} \leq \|\nabla f\|_{X(G)}.$$

K -functionals play an important role in this paper. The K -functional for the pair $(X(\Omega), S_X(\Omega))$ for is defined by

$$(3.1) \quad K(t, f; X(\Omega), S_X(\Omega)) = \inf_{g \in S_X(\Omega)} \{ \|f - g\|_{X(\Omega)} + t \|\nabla g\|_{X(\Omega)} \}.$$

If G is an open subset of Ω , each competing decomposition for the calculation of $K(t, f; X(\Omega), S_X(\Omega))$ produces by restriction a decomposition for the calculation of $f\chi_G$ and we have

$$K(t, f\chi_G; X_r(G), S_{X_r}(G)) \leq K(t, f; X(\Omega), S_X(\Omega)).$$

Notice that from our definition of $S_X(\Omega)$ it does not follow that $h \in S_X$ implies that $h \in X$. However, under mild conditions on X , one can guarantee that $h \in X$. Indeed, using the isoperimetric profile $I = I_{(\Omega, d, \mu)}$, let us define the associated **isoperimetric Hardy operator** by

$$Q_I f(t) = \int_t^{\mu(\Omega)} f(s) \frac{ds}{I(s)} \quad (f \geq 0).$$

Suppose that there exists an absolute constant $c > 0$ such that, for all $f \in \bar{X}$, such that $f \geq 0$, and with $\text{supp}(f) \subset (0, \mu(\Omega)/2)$, we have

$$(3.2) \quad \|Q_I f\|_{\bar{X}} \leq c \|f\|_{\bar{X}}.$$

Then, it was shown in [69] that for all $h \in S_X$,

$$\left\| h - \frac{1}{\mu(\Omega)} \int_{\Omega} h \right\|_X \preceq \|\nabla h\|_X.$$

Therefore, since constant functions belong to X we can then conclude that indeed $h \in X$. It is easy to see that if $\underline{\alpha}_X > 0$, condition (3.2) is satisfied. Indeed, from the concavity of I , it follows that $\frac{I(s)}{s}$ is decreasing, therefore

$$\frac{I(\mu(\Omega)/2)}{\mu(\Omega)/2} \leq \frac{I(s)}{s}, \quad s \in (0, \mu(\Omega)/2).$$

It follows that if $s \in (0, \mu(\Omega)/2)$, then

$$s \leq \frac{\mu(\Omega)/2}{I(\mu(\Omega)/2)} I(s) = c I(s).$$

Consequently, for all $f \geq 0$, with $\text{supp}(f) \subset (0, \mu(\Omega)/2)$,

$$Q_I f(t) = \int_t^{\mu(\Omega)/2} f(s) \frac{ds}{I(s)} \leq c \int_t^{\mu(\Omega)/2} f(s) \frac{ds}{s} = Q f(t).$$

Therefore,

$$\|Q_I f\|_{\bar{X}} \leq c \|Q f\|_{\bar{X}} \leq c_X \|f\|_{\bar{X}},$$

where the last inequality follows from the fact that $\underline{\alpha}_X > 0$. We can avoid placing restrictions on X if instead we impose more conditions on the isoperimetric profile. For example, suppose that the following condition ⁶ on I holds:

$$(3.3) \quad \int_0^{\mu(\Omega)/2} \frac{ds}{I(s)} = c < \infty.$$

⁶A typical example is $I(t) \simeq t^{1-1/n}$, near zero.

Then, for $f \in L^\infty$ we have

$$\begin{aligned} Q_I f(t) &\leq \|f\|_{L^\infty} \int_t^{\mu(\Omega)} \frac{ds}{I(s)} \\ &\leq c \|f\|_{L^\infty} . \end{aligned}$$

Consequently, Q_I is bounded on L^∞ . Since, as we have already seen $Q_I \leq Q$, it follows that Q_I is also bounded on L^1 , and therefore, by Calderón's interpolation theorem, Q_I is bounded on any r.i. space X . In particular, (3.2) is satisfied.

CHAPTER 3

K-functionals and isoperimetry

1. Summary

Let (Ω, d, μ) be a metric measure space satisfying the usual assumptions. In this chapter we show that for every r.i. space $X(\Omega)$ we have that for all $f \in X + S_X$,

$$(1.1) \quad |f|_{\mu}^{**}(t) - |f|_{\mu}^*(t) \leq 2 \frac{K\left(\frac{t}{I_{\Omega}(t)}, f; X, S_X\right)}{\phi_X(t)}, \quad 0 < t < \mu(\Omega).$$

This extends one of the main results of [72]. In Section 3 we prove a variant of inequality (1.1) that will play an important role in Chapter 4, where embeddings into the space of continuous functions will be analyzed. In Section 4 we show the equivalence between (1.1) for $X = L^1$ and the isoperimetric inequality.

2. Estimation of the oscillation in terms of K -functionals

THEOREM 7. *Let X be a r.i. space on Ω . Then (1.1) holds for all $f \in X + S_X$.*

PROOF. The outline of the proof is as follows. As a first step we shall assume that $f \in L^{\infty}$ and prove the weaker inequality: for all $0 < t < \mu(\Omega)$,

$$(2.1) \quad |f|_{\mu}^{**}(t) - |f|_{\mu}^*(t) \leq \frac{1}{\phi_X(t)} \inf_{0 \leq h \leq |f|, h \in S_X} \left\{ \| |f| - h \|_X + \frac{t}{I(t)} \| |\nabla h| \|_X \right\}.$$

Then, using truncations we will show that the right hand side of (2.1) can be indeed estimated by the K -functional: if $f \in L^{\infty}$ then we have,

$$(2.2) \quad \frac{1}{\phi_X(t)} \inf_{0 \leq h \leq |f|, h \in S_X} \left\{ \| |f| - h \|_X + \frac{t}{I(t)} \| |\nabla h| \|_X \right\} \leq 2 \frac{K\left(\frac{t}{I(t)}, f; X, S_X\right)}{\phi_X(t)}.$$

Combining (2.1) with (2.2) we obtain (1.1) for bounded functions. In the final step we extend the result to all functions in $X + S_X$ using further truncation and limiting arguments.

Positivity will play a role in the arguments so it will be useful to note for future use that if $\|\cdot\|$ denotes either $\|\cdot\|_X$ or $\|\cdot\|_{S_X}$, we have

$$(2.3) \quad \| |f| \| \leq \| f \|.$$

Let $\varepsilon > 0$, and consider any decomposition $f = f - h + h$ with $h \in S_X$, such that

$$(2.4) \quad \| f - h \|_X + t \| |\nabla h| \|_X \leq K(t, f; X, S_X) + \varepsilon.$$

This decomposition of f also produces the following decomposition of $|f|$:

$$|f| = |f| - |h| + |h|,$$

where, by (2.3), $|h| \in S_X$. Therefore, by (2.3) and (2.4) we have

$$\begin{aligned} |||f| - |h|||_X + t |||\nabla |h|||_X &\leq \|f - h\|_X + t |||\nabla h|||_X \\ &\leq K(t, f; X, S_X) + \varepsilon. \end{aligned}$$

Consequently,

$$(2.5) \quad \inf_{0 \leq h \in S_X} \{ |||f| - h||_X + t |||\nabla h|||_X \} \leq K(t, f; X, S_X).$$

In fact, we can now improve (2.5) as follows

$$(2.6) \quad \inf_{0 \leq h \leq |f|, h \in S_X} \{ |||f| - h||_X + t |||\nabla h|||_X \} \leq 2 \inf_{0 \leq h \in S_X} \{ |||f| - h||_X + t |||\nabla h|||_X \}.$$

To see this we use our current assumption that f is bounded. For $0 \leq h \in S_X$, let

$$g = \min(h, \|f\|_\infty) \in S_X.$$

Then,

$$|\nabla g| \leq |\nabla h|,$$

and, moreover, we have

$$\begin{aligned} |||f| - g||_X + t |||\nabla g|||_X &\leq \left\| (|f| - h) \chi_{\{h \leq \|f\|_\infty\}} \right\|_X + \left\| (|f| - \|f\|_\infty) \chi_{\{h > \|f\|_\infty\}} \right\|_X \\ &\quad + t |||\nabla h|||_X \\ &\leq |||f| - h||_X + \left\| (|f| - \|f\|_\infty) \chi_{\{h > \|f\|_\infty\}} \right\|_X + t |||\nabla h|||_X. \end{aligned}$$

Now, since

$$\left| (|f| - \|f\|_\infty) \chi_{\{h > \|f\|_\infty\}} \right| = (\|f\|_\infty - |f|) \chi_{\{h > \|f\|_\infty\}} \leq (h - |f|) \chi_{\{h > \|f\|_\infty\}},$$

we see that

$$\left\| (|f| - \|f\|_\infty) \chi_{\{h > \|f\|_\infty\}} \right\|_X = \left\| (h - |f|) \chi_{\{h > \|f\|_\infty\}} \right\|_X \leq |||f| - h||_X,$$

and (2.6) follows. Combining (2.5) and (2.6), we derive (2.2).

To conclude the first part of the proof it remains to establish (2.1). Let $f \in X + S_X$, we need to estimate $|f|_\mu^{**}(t) - |f|_\mu^*(t)$. Let $0 \leq h \in S_X$ be such that $h \leq |f|$. Observe that for any decomposition $|f| = f_0 + f_1$ with f_i positive, $i = 0, 1$, we always have $|f|_\mu^{**}(t) \leq |f_0|_\mu^{**}(t) + |f_1|_\mu^{**}(t)$, with $|f_i|_\mu^*(t) \leq |f|_\mu^*(t)$. Therefore, writing $|f| = (|f| - h) + h$, we have

$$|f|_\mu^{**}(t) \leq ||f| - h||_\mu^{**}(t) + |h|_\mu^{**}(t),$$

and

$$|h|_\mu^*(t) \leq |f|_\mu^*(t).$$

Consequently,

$$\begin{aligned} |f|_\mu^{**}(t) - |f|_\mu^*(t) &\leq ||f| - h||_\mu^{**}(t) + |h|_\mu^{**}(t) - |f|_\mu^*(t) \\ &\leq ||f| - h||_\mu^{**}(t) + |h|_\mu^{**}(t) - |h|_\mu^*(t) \\ &\leq ||f| - h||_\mu^{**}(t) + \frac{t}{I(t)} |\nabla h|_\mu^{**}(t) \text{ (by [(1.3), Chapter 2])} \\ (2.7) \quad &= A(t) + B(t). \end{aligned}$$

We now estimate the two terms on the right hand side of (2.7). For the term $A(t)$: note that for any $g \in X$,

$$|g|_{\mu}^{**}(t) = \frac{1}{t} \int_0^t |g|_{\mu}^*(s) ds = \frac{1}{t} \int_0^1 |g|_{\mu}^*(s) \chi_{(0,t)}(s) ds.$$

Therefore, by Hölder's inequality, [(2.1) and (2.2) of Chapter2], we have

$$\begin{aligned} \|f| - h|_{\mu}^{**}(t) &= \frac{1}{t} \int_0^1 \|f| - h|_{\mu}^*(s) \chi_{(0,t)}(s) ds \\ &\leq \|(|f| - h)\|_X \frac{\phi_{X'}(t)}{t} \\ (2.8) \quad &= \|(|f| - h)\|_X \frac{1}{\phi_X(t)}. \end{aligned}$$

Similarly, for $B(t)$ we get

$$(2.9) \quad B(t) = \frac{t}{I(t)} |\nabla h|_{\mu}^{**}(t) \leq \frac{t}{I(t)} \frac{\|\nabla h\|_X}{\phi_X(t)}.$$

Inserting (2.8) and (2.9) back in (2.7) we find that,

$$|f|_{\mu}^{**}(t) - |f|_{\mu}^*(t) \leq \frac{1}{\phi_X(t)} \left(\| |f| - h \|_X + \frac{t}{I(t)} \|\nabla h\|_X \right).$$

Taking infimum, we thus see that (2.1) holds and consequently (1.1) holds for bounded functions. We now consider the case when f is not bounded. For $n \in \mathbb{N}$, let $f_n = \min(|f|, n)$. By the first part of the proof we have,

$$|f_n|_{\mu}^{**}(t) - |f_n|_{\mu}^*(t) \leq \frac{2}{\phi_X(t)} \inf_{0 \leq h \in S_X} \left\{ \| |f_n| - h \|_X + \frac{t}{I(t)} \|\nabla h\|_X \right\}.$$

For $0 \leq h \in S_X$, let $h_n = \min(h, n)$. It follows that

$$\begin{aligned} |f_n|_{\mu}^{**}(t) - |f_n|_{\mu}^*(t) &\leq \frac{2}{\phi_X(t)} \left\{ \| |f_n| - h_n \|_X + \frac{t}{I(t)} \|\nabla h_n\|_X \right\} \\ &\leq \frac{2}{\phi_X(t)} \left\{ \| |f| - h \|_X + \frac{t}{I(t)} \|\nabla h\|_X \right\}. \end{aligned}$$

Therefore, taking infimum we have

$$|f_n|_{\mu}^{**}(t) - |f_n|_{\mu}^*(t) \leq \frac{2}{\phi_X(t)} K\left(\frac{t}{I(t)}, f; X, S_X\right).$$

Since $f_n = \min(|f|, n) \nearrow |f|$, we have (cf. [13])

$$|f_n|_{\mu}^{**}(t) - |f_n|_{\mu}^*(t) \xrightarrow{n \rightarrow \infty} |f|_{\mu}^{**}(t) - |f|_{\mu}^*(t),$$

and we see that

$$|f|_{\mu}^{**}(t) - |f|_{\mu}^*(t) \leq \frac{2}{\phi_X(t)} K\left(\frac{t}{I(t)}, f; X, S_X\right),$$

as we wished to show. \square

A useful variant of the previous result can be stated as follows (cf. [72] for a somewhat weaker result).

THEOREM 8. *Let $X = X(\Omega)$ be a r.i. space with $\underline{\alpha}_X > 0$. Then, there exists a constant $c = c(X) > 0$ such that, for all $f \in X$, we have*

$$(2.10) \quad \left\| \left(|f|_\mu^*(\cdot) - |f|_\mu^*(t) \right) \chi_{(0,t)}(\cdot) \right\|_{\bar{X}} \leq cK \left(\frac{t}{I(t)}, f; X, S_X \right), \quad 0 < t < \mu(\Omega).$$

PROOF. Fix $t \in (0, \mu(\Omega))$ and assume that f is bounded. Let $h \in Lip(\Omega)$ be such that $h \leq |f|$, and fix $g \in \bar{X}'$ with $\|g\|_{\bar{X}'} = 1$. Recall that \bar{X}' is a r.i. space on $([0, \mu(\Omega)], m)$, where m denotes Lebesgue measure. We let $g^* := g_m^*$. Consider the decomposition

$$|f| = (|f| - h) + h.$$

Then, using [(1.1), Chapter 2], we have that

$$(2.11) \quad \begin{aligned} & \int_0^t (|f|_\mu^*(s) - |f|_\mu^*(t)) \chi_{(0,t)}(s) |g|^*(s) ds \\ &= \int_0^t |f|_\mu^*(s) |g|^*(s) ds - |f|_\mu^*(t) \int_0^t |g|^*(s) ds \\ &= \int_0^t |f|_\mu^*(s) |g|^*(s) ds - |f|_\mu^*(t) \int_0^t |g|^*(s) ds \\ &\leq \int_0^t ||f| - h|_\mu^*(s) |g|^*(s) ds + \int_0^t |h|_\mu^*(s) |g|^*(s) ds - |h|_\mu^*(t) \int_0^t |g|^*(s) ds \\ &= \int_0^t ||f| - h|_\mu^*(s) |g|^*(s) ds + \int_0^t (|h|_\mu^*(s) - |h|_\mu^*(t)) |g|^*(s) ds \\ &\leq \int_0^{\mu(\Omega)} ||f| - h|_\mu^*(s) |g|^*(s) ds + \int_0^{\mu(\Omega)} (|h|_\mu^*(s) - |h|_\mu^*(t)) \chi_{(0,t)}(s) |g|^*(s) ds \\ &\leq |||f| - h||_{\bar{X}} + \left\| (|h|_\mu^*(\cdot) - |h|_\mu^*(t)) \chi_{(0,t)} \right\|_{\bar{X}}. \end{aligned}$$

Let $(h_n)_n$ be the sequence associated to h that is provided by Lemma 2. Then, taking in account the concavity of the isoperimetric profile I , we get

$$\begin{aligned} |h_n|_\mu^*(s) - |h_n|_\mu^*(t) &= \int_s^t \left(-|h_n|_\mu^* \right)'(r) dr \\ &= \int_s^t \left(-|h_n|_\mu^* \right)'(r) I(r) \frac{dr}{I(r)} \\ &\leq \frac{t}{I(t)} \int_s^t \left(-|h_n|_\mu^* \right)'(r) I(r) \frac{dr}{r} \\ &\leq \frac{t}{I(t)} \int_s^{\mu(\Omega)} \left(-|h_n|_\mu^* \right)'(r) I(r) \frac{dr}{r}. \end{aligned}$$

Using the fact that $\underline{\alpha}_X > 0$ and $t/I(t)$ is increasing, we see that

$$\begin{aligned} \left\| (|h_n|_\mu^*(\cdot) - |h_n|_\mu^*(t)) \chi_{(0,t)}(\cdot) \right\|_{\bar{X}} &\leq c \frac{t}{I(t)} \left\| \left(-|h_n|_\mu^* \right)'(\cdot) I(\cdot) \right\|_{\bar{X}} \\ &\leq c \frac{t}{I(t)} \|\nabla h_n\|_X \quad (\text{by [(1.6), Chapter 2]}) \\ &\leq c \frac{t}{I(t)} \left(1 + \frac{1}{n} \right) \|\nabla h\|_{\bar{X}} \quad (\text{by [(1.4), Chapter 2]}). \end{aligned}$$

On the other hand, by [(1.5) Chapter 2], $h_n \xrightarrow{n \rightarrow \infty} h$ in L^1 , by Lemma 1 we get $|h_n|_\mu^*(s) \rightarrow |h|_\mu^*(s)$, and we find that

$$(2.12) \quad \begin{aligned} \left\| \left(|h|_\mu^*(\cdot) - |h|_\mu^*(t) \right) \chi_{(0,t)}(\cdot) \right\|_{\bar{X}} &= \lim_{n \rightarrow \infty} \left\| \left(|h_n|_\mu^*(\cdot) - |h_n|_\mu^*(t) \right) \chi_{(0,t)}(s) \right\|_{\bar{X}} \\ &\leq c \frac{t}{I(t)} \|\nabla h\|_X. \end{aligned}$$

Combining (2.11) and (2.12) we get

$$(2.13) \quad \begin{aligned} &\int_0^{\mu(\Omega)} (|f|_\mu^*(s) - |f|_\mu^*(t)) \chi_{(0,t)}(s) |g|_\mu^*(s) ds \\ &\leq c(\|f| - h\|_X + \frac{t}{I(t)} \|\nabla h\|_X). \end{aligned}$$

Since $(|f|_\mu^*(s) - |f|_\mu^*(t)) \chi_{(0,t)}(s)$ is a decreasing function of s , we have

$$\left(|f|_\mu^*(s) - |f|_\mu^*(t) \right) \chi_{(0,t)}(s) = \left(\left(|f|_\mu^*(\cdot) - |f|_\mu^*(t) \right) \chi_{(0,t)}(\cdot) \right)^*(s).$$

Combining successively duality, the last formula and (2.13), we obtain

$$\begin{aligned} \left\| \left(|f|_\mu^*(s) - |f|_\mu^*(t) \right) \chi_{(0,t)}(s) \right\|_{\bar{X}} &= \sup_{\|g\|_{\bar{X}'} \leq 1} \int_0^{\mu(\Omega)} \left[\left(|f|_\mu^*(\cdot) - |f|_\mu^*(t) \right) \chi_{(0,t)}(\cdot) \right]^*(s) |g|_\mu^*(s) ds \\ &= \sup_{\|g\|_{\bar{X}'} \leq 1} \int_0^{\mu(\Omega)} \left(|f|_\mu^*(s) - |f|_\mu^*(t) \right) \chi_{(0,t)}(s) |g|_\mu^*(s) ds \\ &\leq c \left(\|f| - h\|_X + \frac{t}{I(t)} \|\nabla h\|_X \right), \end{aligned}$$

where c is an absolute constant. Consequently, if f is bounded there exists an absolute constant such that

$$\begin{aligned} \left\| \left(|f|_\mu^*(s) - |f|_\mu^*(t) \right) \chi_{(0,t)}(s) \right\|_{\bar{X}} &\leq c \inf_{0 \leq h \leq |f|, h \in S_X} \left\{ \|f| - h\|_X + \frac{t}{I(t)} \|\nabla h\|_X \right\} \\ &\leq cK \left(\frac{t}{I(t)}, f; X, S_X \right) \text{ (by (2.2)).} \end{aligned}$$

Suppose now that f is not bounded. Let $f_n = \min(|f|, n)$, then, as in the proof of Theorem 7, we see that

$$\left\| \left(|f_n|_\mu^*(\cdot) - |f_n|_\mu^*(t) \right) \chi_{(0,t)}(\cdot) \right\|_X \leq cK \left(\frac{t}{I(t)}, f; X, S_X \right).$$

Taking limits using the Fatou property of X , we finally obtain

$$\left\| \left(|f|_\mu^*(s) - |f|_\mu^*(t) \right) \chi_{(0,t)}(s) \right\|_{\bar{X}} \leq cK \left(\frac{t}{I(t)}, f; X, S_X \right).$$

□

REMARK 3. *For perspective, we now show that (2.10) can in turn be used to give a direct proof of the fact that for all $f \in X$,*

$$(2.14) \quad f_\mu^{**}(t) - f_\mu^*(t) \leq c \frac{K \left(\frac{t}{I(t)}, f; X, S_X \right)}{\phi_X(t)}, \quad 0 < t < \mu(\Omega).$$

However, since we shall use Theorem 8, the constant we obtain depends on the space X .

PROOF. We have,

$$\begin{aligned}
 |f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t) &= \frac{1}{t} \int_0^t \left(|f|_{\mu}^{*}(s) - |f|_{\mu}^{*}(t) \right) \chi_{(0,t)}(s) \\
 &\leq \left\| \left(|f|_{\mu}^{*}(s) - |f|_{\mu}^{*}(t) \right) \chi_{(0,t)}(s) \right\|_{\bar{X}} \frac{\phi_{X'}(t)}{t} \quad (\text{by Hölder's inequality}) \\
 &\leq cK \left(\frac{t}{I(t)}, f; X, S_X \right) \frac{\phi_{X'}(t)}{t} \\
 &= c \frac{K \left(\frac{t}{I(t)}, f; X, S_X \right)}{\phi_X(t)}.
 \end{aligned}$$

□

EXAMPLE 1. For familiar spaces (2.10) takes a more concrete form. For example, if $X = L^p$, $1 \leq p < \infty$, then (2.10) becomes

$$(2.15) \quad \int_0^t \left(|f|_{\mu}^{*}(s) - |f|_{\mu}^{*}(t) \right)^p ds \leq c_p \left(K \left(\frac{t}{I(t)}, f; L^p, S_{L^p} \right) \right)^p.$$

In particular, when $p = 1$, the left hand side of (2.15) becomes

$$t(|f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t)) = \int_0^t (f_{\mu}^{*}(s) - f_{\mu}^{*}(t)) ds.$$

As a consequence, (2.14) and (2.10) represent the same inequality when $X = L^1$.

The next easy variant of Theorem 7 gives more flexibility for some applications.

THEOREM 9. Let X and Y be a r.i. spaces on Ω . Then, for each $f \in X + S_Y$ we have

$$(2.16) \quad |f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t) \leq 2 \frac{K \left(\frac{t}{I(t)} \frac{\phi_X(t)}{\phi_Y(t)}, f; X, S_Y \right)}{\phi_X(t)}, \quad 0 < t < \mu(\Omega).$$

PROOF. Since the proof its almost the same as the proof of Theorem 7 we shall only briefly indicate the necessary changes. Let $f = f_0 + f_1$ be a decomposition of f , using estimate (2.9), with Y instead of X , we get

$$|\nabla f_1|_{\mu}^{**}(t) \leq \frac{\|\nabla f_1\|_Y}{\phi_Y(t)}.$$

Therefore, writing

$$(2.17) \quad \frac{t}{I(t)} |\nabla f_1|_{\mu}^{**}(t) \leq \frac{\phi_X(t)}{\phi_X(t)} \frac{t}{I(t)} \frac{\|\nabla f_1\|_Y}{\phi_Y(t)},$$

and inserting (2.8) and (2.17) back in (2.7) we find that, for $0 < t < \mu(\Omega)$,

$$\begin{aligned}
 |f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t) &\leq \frac{\|f_0\|_X}{\phi_X(t)} + \frac{\phi_X(t)}{\phi_X(t)} \frac{t}{I(t)} \frac{\|\nabla f_1\|_Y}{\phi_Y(t)} \\
 &\leq \frac{1}{\phi_X(t)} \left(\|f_0\|_X + \phi_X(t) \frac{t}{I(t)} \frac{\|\nabla f_1\|_Y}{\phi_Y(t)} \right).
 \end{aligned}$$

□

REMARK 4. Obviously if there exists a constant $c > 0$ such that for all t

$$(2.18) \quad \phi_X(t) \leq c\phi_Y(t),$$

then for each $f \in X + w_Y$,

$$|f|_{\mu}^{**}(t) - |f|_{\mu}^*(t) \leq 2 \frac{K\left(\frac{ct}{I(t)}, f; X, w_Y\right)}{\phi_X(t)}, \quad 0 < t < \mu(\Omega).$$

Note that $Y \subset X$ implies that (2.18) holds.

3. A variant of Theorem 7

This section is devoted to the proof of an improvement of Theorem 7 that will play an important role in Chapter 4.

THEOREM 10. Let X be a r.i. space on Ω . Then, for each $f \in X + S_X$,

$$(3.1) \quad |f|_{\mu}^{**}(t) - |f|_{\mu}^*(t) \leq 2 \frac{K(\psi(t), f; X, S_X)}{\phi_X(t)}, \quad 0 < t < \mu(\Omega),$$

where

$$(3.2) \quad \psi(t) = \frac{\phi_X(t)}{t} \left\| \frac{s}{I(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}.$$

Before we proceed with the proof let us show that, indeed, (3.1) is stronger than (1.1). In fact, the function ψ defined in (3.2) is smaller or equal than $\frac{t}{I(t)}$. This follows readily from the fact that $\frac{s}{I(s)}$ is increasing,

$$\begin{aligned} \psi(t) &= \frac{\phi_X(t)}{t} \left\| \frac{s}{I(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'} \\ &\leq \frac{\phi_X(t)}{t} \frac{t}{I(t)} \left\| \chi_{(0,t)}(s) \right\|_{\bar{X}'} \\ &= \frac{\phi_X(t)}{t} \frac{t}{I(t)} \phi_{X'}(t) \\ &= \frac{t}{I(t)}. \end{aligned}$$

PROOF. (of Theorem 10) The proof follows very closely the proof of Theorem 7 therefore we shall only indicate the necessary changes. Let $f \in X + S_X$, and let $0 \leq h \in S_X$ be such that $h \leq |f|$. Proceeding as before we find

$$|f|_{\mu}^{**}(t) \leq ||f| - h|_{\mu}^{**}(t) + |h|_{\mu}^{**}(t).$$

Therefore,

$$|f|_{\mu}^{**}(t) - |f|_{\mu}^*(t) \leq ||f| - h|_{\mu}^{**}(t) + |h|_{\mu}^{**}(t) - |h|_{\mu}^*(t).$$

The first term to the right was estimated in (2.8) by $\|(|f| - h)\|_X \frac{1}{\phi_X(t)}$. To estimate $|h|_{\mu}^{**}(t) - |h|_{\mu}^*(t)$, consider $(h_n)_n$ the sequence of Lip functions associated to h

provided by Lemma 2. Then

$$\begin{aligned}
|h_n|_\mu^{**}(s) - |h_n|_\mu^*(t) &= \frac{1}{t} \int_0^t s \left(-|h_n|_\mu^* \right)'(s) ds \\
&= \frac{1}{t} \int_0^t s \left(-|h_n|_\mu^* \right)'(s) I(s) \frac{ds}{I(s)} \\
&\leq \frac{1}{t} \left\| \frac{s}{I(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'} \left\| \left(-|h_n|_\mu^* \right)'(s) I(s) \right\|_X \quad (\text{by Hölder's inequality}) \\
&\leq \frac{1}{t} \left\| \frac{s}{I(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'} \|\nabla h_n\|_X \quad (\text{by [(1.6), Chapter 2]}) \\
&\leq \frac{1}{t} \left\| \frac{s}{I(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'} \left(1 + \frac{1}{n} \right) \|\nabla h\|_{\bar{X}} \quad (\text{by [(1.4), Chapter 2]})
\end{aligned}$$

On the other hand, from [(1.5) Chapter 2] and Lemma 1, we get $|h_n|_\mu^*(s) \rightarrow |h|_\mu^*(s)$, and $|h_n|_\mu^{**}(s) \rightarrow |h|_\mu^{**}(s)$. Consequently,

$$|h|_\mu^{**}(t) - |h|_\mu^*(t) \leq \frac{1}{t} \left\| \frac{s}{I(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'} \|\nabla h\|_X.$$

It follows that

$$\begin{aligned}
|f|_\mu^{**}(t) - |f|_\mu^*(t) &\leq \inf_{0 \leq h \leq |f|, h \in S_X} \left\{ \frac{\| |f| - h \|_X}{\phi_X(t)} + \frac{1}{t} \left\| \frac{s}{I(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'} \|\nabla h\|_X \right\} \\
&= \frac{1}{\phi_X(t)} \inf_{0 \leq h \leq |f|, h \in S_X} \{ \| |f| - h \|_X + \psi(t) \|\nabla h\|_X \} \\
&\leq 2 \frac{K(\psi(t), f; X, S_X)}{\phi_X(t)} \quad (\text{by (2.2)}).
\end{aligned}$$

If f is not bounded, we use a familiar truncation argument. Let $f_n = \min(|f|, n)$, then as before, for any $0 \leq h \in S_X$,

$$\begin{aligned}
|f_n|_\mu^{**}(s) - |f_n|_\mu^*(t) &\leq \frac{2}{\phi_X(t)} \left(\|f_n - h\|_X + \frac{1}{t} \left\| \frac{s}{I(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'} \|\nabla h\|_X \right) \\
&\leq \frac{2}{\phi_X(t)} \left(\| |f| - h \|_X + \frac{1}{t} \left\| \frac{s}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'} \|\nabla h\|_X \right).
\end{aligned}$$

Letting $n \rightarrow \infty$, (3.1) follows. \square

4. Isoperimetry

In this section we show the connection of (1.1) and (3.1) with isoperimetry. Observe that (1.1) holds for all r.i. spaces. In particular, it holds for $X = L^1$.

THEOREM 11. *The following are equivalent*

i) *Isoperimetric inequality: There exists an absolute constant $c > 0$, such that*

$$I(\mu(A)) \leq cP(A; \Omega), \text{ for all Borel sets } A \text{ with } 0 < \mu(A) < \mu(\Omega).$$

ii) *There exists an absolute constant $c > 0$ such that for each $f \in L^1$,*

$$(4.1) \quad |f|_\mu^{**}(t) - |f|_\mu^*(t) \leq c \frac{K\left(\frac{t}{I(t)}, f; L^1, S_{L^1}\right)}{t}, \quad 0 < t < \mu(\Omega).$$

PROOF. We only need to prove $ii) \rightarrow i)$. Let A be a Borel set with $0 < \mu(A) < \mu(\Omega)$. We may assume, without loss, that $P(A; \Omega) < \infty$. By [18, Lemma 3.7] we can select a sequence $\{f_n\}_{n \in \mathbb{N}}$ of Lip functions such that $f_n \xrightarrow{L^1} \chi_A$, and

$$P(A; \Omega) \geq \limsup_{n \rightarrow \infty} \|\nabla f_n\|_{L^1}.$$

From (4.1) we know that there exists a constant $c > 0$ such that for all $0 < t < \mu(\Omega)$,

$$|f_n|_{\mu}^{**}(s) - |f_n|_{\mu}^*(t) \leq 2 \frac{K\left(\frac{t}{I(t)}, f_n; L^1, S_{L^1}\right)}{t}.$$

We take limits when $n \rightarrow \infty$ on both sides of this inequality. To compute the left hand side we observe that, since, $f_n \xrightarrow{L^1} \chi_A$, Lemma 1 implies that:

$$\begin{aligned} |f_n|_{\mu}^{**}(t) &\rightarrow |\chi_A|_{\mu}^{**}(t), \text{ uniformly for } t \in [0, 1], \text{ and} \\ |f_n|_{\mu}^*(t) &\rightarrow |\chi_A|_{\mu}^*(t) \text{ for } t \in (0, 1). \end{aligned}$$

Fix $t > \mu(A)$. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(|f_n|_{\mu}^{**}(s) - |f_n|_{\mu}^*(t) \right) &= (\chi_A)_{\mu}^{**}(t) - (\chi_A)_{\mu}^*(t) \\ &= (\chi_A)_{\mu}^{**}(t) \text{ (since } (\chi_A)_{\mu}^*(t) = \chi_{(0, \mu(A))}(t) = 0) \\ &= \frac{\mu(A)}{t}. \end{aligned}$$

Now, to estimate the right hand side we observe that, for each n , $f_n \in L^1 \cap S_{L^1}$. Consequently, by the definition of K functional, we have

$$\begin{aligned} \frac{K\left(\frac{t}{I(t)}, f_n, L^1, S_{L^1}\right)}{t} &\leq \min \left\{ \frac{\|f_n\|_{L^1}}{t}, \frac{1}{I(t)} \|\nabla f_n\|_{L^1} \right\} \\ &\leq \min \left\{ \frac{\|f_n\|_{L^1}}{t}, \frac{1}{I(t)} P(A; \Omega) \right\}. \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} \frac{K\left(\frac{t}{I(t)}, f_n, L^1, S_{L^1}\right)}{t} \leq \min \left\{ \frac{\mu(A)}{t}, \frac{1}{I(t)} P(A; \Omega) \right\}.$$

Combining these estimates we find that for all $t > \mu(A)$,

$$\frac{\mu(A)}{t} \leq 2 \frac{1}{I(t)} P(A; \Omega).$$

Therefore, letting $t \rightarrow \mu(A)$ we find

$$1 \leq 2 \frac{1}{I(\mu(A))} P(A; \Omega).$$

or

$$I(\mu(A)) \leq 2P(A; \Omega),$$

as we wished to show. \square

CHAPTER 4

Embedding into continuous functions

1. Introduction and Summary

In this chapter we obtain a general version of the Morrey-Sobolev theorem on metric measure spaces (Ω, d, μ) satisfying the usual assumptions.

1.1. Inequalities for signed rearrangements. Let (Ω, d, μ) be a metric measure space satisfying the usual assumptions. We collect a few more facts about signed rearrangements. First let us note that for $c \in \mathbb{R}$,

$$(1.1) \quad (f + c)_\mu^*(t) = f_\mu^*(t) + c.$$

Moreover, if $X(\Omega)$ is a r.i. space, we have

$$\| |f|_\mu^* \|_{\bar{X}(0, \mu(\Omega))} = \| |f| \|_{X(\Omega)} = \| f \|_{X(\Omega)} = \| f_\mu^* \|_{\bar{X}(0, \mu(\Omega))},$$

where $\bar{X}(0, \mu(\Omega))$ is the representation space of $X(\Omega)$.

The results of the previous chapter can be easily formulated in terms of signed rearrangements of f . In particular, we shall now discuss in detail the following extension (variant) of Theorem 10.

THEOREM 12. *Let X be a r.i. space on Ω . Then, for each $f \in X + S_X$, we have*

$$(1.2) \quad f_\mu^{**}(t) - f_\mu^*(t) \leq 2 \frac{K(\psi(t), f; X, S_X)}{\phi_X(t)}, \quad 0 < t < \mu(\Omega),$$

where

$$(1.3) \quad \psi(t) = \frac{\phi_X(t)}{t} \left\| \frac{s}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}.$$

PROOF. Let us first further assume that f is bounded from below, and let $c = \inf_\Omega f$. Then, since $f - c \geq 0$, we can apply Theorem 10 to $f - c$, and we obtain

$$(1.4) \quad (f - c)_\mu^{**}(t) - (f - c)_\mu^*(t) \leq 2 \frac{K(\psi(t), f - c; X, S_X)}{\phi_X(t)}.$$

We can simplify the left hand side of (1.4) noting that by (1.1)

$$f_\mu^{**}(t) - f_\mu^*(t) = (f - c)_\mu^{**}(t) - (f - c)_\mu^*(t).$$

On the other hand, the sub-additivity of the K -functional, and the fact that for a constant function c we have $K(t, c; X, S_X) = 0$, allow us to estimate the right hand side of (1.4) as follows

$$\begin{aligned} K(\psi(t), f - c; X, S_X) &\leq K(\psi(t), f; X, S_X) + K(\psi(t), c; X, S_X) \\ &= K(\psi(t), f; X, S_X). \end{aligned}$$

Combining estimates, we get

$$f_\mu^{**}(t) - f_\mu^*(t) \leq 2 \frac{K(\psi(t), f; X, S_X)}{\phi_X(t)}.$$

If f is not bounded from below, we use an approximation argument. Let

$$f_n = \max(f, -n), \quad n = 1, 2, \dots$$

The previous discussion gives

$$(f_n)_\mu^{**}(t) - (f_n)_\mu^*(t) \leq 2 \frac{K(\psi(t), f_n; X, S_X)}{\phi_X(t)}.$$

We now take limits. To compute the left hand side we observe that $f_n(x) \rightarrow f(x)$ μ -a.e., and $|f_n| \leq |f|$, implies (cf. [13])

$$(f_n)_\mu^{**}(t) - (f_n)_\mu^*(t) \xrightarrow{n \rightarrow \infty} f_\mu^{**}(t) - f_\mu^*(t), \quad 0 < t < \mu(\Omega).$$

We estimate the right hand side as follows. Given $\varepsilon > 0$, select $h^\varepsilon \in S_X$ such that

$$(1.5) \quad \|f - h^\varepsilon\|_X + \psi(t) \|\nabla h^\varepsilon\|_X \leq K(\psi(t), f; X, S_X) + \varepsilon.$$

Define $h_n^\varepsilon = \max(h^\varepsilon, -n)$, then

$$(1.6) \quad h_n^\varepsilon \in S_X \quad \text{with} \quad |\nabla h_n^\varepsilon| \leq |\nabla h^\varepsilon|.$$

By a straightforward analysis of all possible cases we see that

$$(1.7) \quad \|f_n - h_n^\varepsilon\|_X \leq \|f - h^\varepsilon\|_X.$$

Therefore, combining (1.5), (1.6) and (1.7), we obtain

$$K(\psi(t), f_n; X, S_X) \leq K(\psi(t), f; X, S_X),$$

and (1.2) follows. \square

2. Continuity via rearrangement inequalities

In this section we consider the following problems: Characterize in terms of K -functional conditions the functions in $f \in X(\Omega) + S_X(\Omega)$ that are bounded, or essentially continuous. One can rephrase these questions as suitable embedding theorems for Besov type spaces.

We consider boundedness first.

LEMMA 3. *Let (Ω, d, μ) be a metric measure space, and let $X(\Omega)$ be a r.i. space. Assume that the isoperimetric profile I_Ω of (Ω, d, μ) satisfies¹*

$$\left\| \frac{1}{I_\Omega(s)} \right\|_{\bar{X}'} < \infty.$$

Suppose that $f \in X + S_X$ is such that

$$\int_0^{\mu(\Omega)} \frac{K\left(\phi_X(t) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f; X, S_X\right) dt}{\phi_X(t)} \frac{1}{t} < \infty,$$

then f is essentially bounded.

¹This condition is connected with the embedding (see Chapter 5 below)

$$S_X^1 \subset L^\infty.$$

PROOF. By Theorem 10, we have

$$|f|_{\mu}^{**}(t) - |f|_{\mu}^*(t) \leq 2 \frac{K\left(\frac{\phi_X(t)}{t} \left\| \frac{s}{I_{\Omega}(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f; X, S_X\right)}{\phi_X(t)}, \quad 0 < t < \mu(\Omega).$$

Integrating both sides of this inequality from 0 to $\mu(\Omega)$, we find

$$\begin{aligned} \operatorname{ess\,sup}_{\Omega} |f| - \frac{1}{\mu(\Omega)} \int_0^{\mu(\Omega)} |f|_{\mu}^*(s) ds &= f_{\mu}^{**}(0) - f_{\mu}^{**}(\mu(\Omega)) \\ &= \int_0^{\mu(\Omega)} (f_{\mu}^{**}(t) - f_{\mu}^*(t)) \frac{dt}{t} \\ &\leq 2 \int_0^{\mu(\Omega)} \frac{K\left(\frac{\phi_X(t)}{t} \left\| \frac{s}{I_{\Omega}(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f; X, S_X\right)}{\phi_X(t)} \frac{dt}{t} \\ &\leq 2 \int_0^{\mu(\Omega)} \frac{K\left(\phi_X(t) \left\| \frac{1}{I_{\Omega}(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f; X, S_X\right)}{\phi_X(t)} \frac{dt}{t}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|f\|_{L^{\infty}} &\leq \frac{1}{\mu(\Omega)} \int_0^{\mu(\Omega)} |f|_{\mu}^*(s) ds + 2 \int_0^{\mu(\Omega)} \frac{K\left(\phi_X(t) \left\| \frac{1}{I_{\Omega}(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f; X, S_X\right)}{\phi_X(t)} \frac{dt}{t} \\ &\leq \frac{1}{\phi_X(\mu(\Omega))} \|f\|_X + 2 \int_0^{\mu(\Omega)} \frac{K\left(\phi_X(t) \left\| \frac{1}{I_{\Omega}(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f; X, S_X\right)}{\phi_X(t)} \frac{dt}{t}. \end{aligned}$$

□

We shall now consider essential continuity. It will be useful to introduce some notation. Recall (cf. Section 2, Chapter 2) that $X_r(G)$, the restriction of $X(\Omega)$ to G , is defined by

$$X_r(G) = \left\{ u\chi_G : u \in X(\Omega), \quad \|u\|_{X_r(G)} := \|u\chi_G\|_{X(\Omega)} < \infty \right\}.$$

Likewise $S_{X_r}(G)$ is the function space

$$S_{X_r}(G) = \left\{ u\chi_G : u \in S_X(\Omega), \quad \|\nabla(u\chi_G)\|_{X_r(G)} < \infty \right\}.$$

When the open set G is understood from the context, we shall simply write X_r and S_{X_r} . We shall denote by $\bar{X}_r = \bar{X}_r(0, \mu(G))$ the representation space of X_r , and we let X'_r denote the corresponding associated space of X_r (For more information see Chapter 2 Section 2 - (2.9)).

If $f \in X(\Omega) + S_X(\Omega)$, then we obviously have that $f\chi_G \in X_r(G) + S_{X_r}(G)$. However, we can not apply our fundamental inequalities [(1.1), Chapter 3] or (1.3)] since we are now working in the metric space $(G, d|_G, \mu|_G)$ and therefore the isoperimetric profile has changed.

Given $G \subset \Omega$ an open subset, and let $A \subset G$. The **perimeter** of A **relative** to G is defined by

$$P(A; G) = \liminf_{h \rightarrow 0} \frac{\mu(A_h) - \mu(A)}{h},$$

where $A_h = \{x \in G : d(x, A) < h\}$. Obviously

$$P(A; G) \leq P(A; \Omega).$$

The **relative isoperimetric profile** of $G \subset \Omega$ is defined by (see for example [4] and the references quoted therein)

$$I_G(s) = I_{(G,d,\mu)}(s) = \inf \{P(A; G) : A \subset G, \mu(A) = s\}, \quad 0 < s < \mu(G).$$

We say that an **isoperimetric inequality relative** to G holds true if there exists a positive constant C_G such that

$$I_G(s) \geq C_G \min(I_\Omega(s), I_\Omega(\mu(G) - s)) = J_G(t), \quad 0 < s < \mu(G),$$

where I_Ω is the isoperimetric profile of (Ω, d, μ) . Notice that, if $\mu(G) \leq \mu(\Omega)/2$, then $J_G : [0, \mu(G)] \rightarrow [0, \infty)$ is increasing on $(0, \mu(G)/2)$, symmetric about the point $\mu(G)/2$, with $J_G(0) = 0$, and such that

$$I_G \geq J_G,$$

i.e. J_G is an isoperimetric estimator for the metric space $(G, d|_G, \mu|_G)$.

DEFINITION 1. *We will say that a metric measure space (Ω, d, μ) has the **relative isoperimetric property** if for any $x \in \Omega$, there exists a positive number $\delta = \delta(x)$ and a constant C such that for any open ball $B_\alpha(x)$ centered on x with $\mu(B_\alpha(x)) = \alpha$ ($0 < \alpha < \delta$), its **relative isoperimetric profile** $I_{B_\alpha(x)}$ satisfies:*

$$I_{B_\alpha(x)}(s) \geq C \min(I_\Omega(s), I_\Omega(\alpha - s)), \quad 0 < s < \alpha.$$

The following proposition will be useful in what follows:

PROPOSITION 2. *Let I_Ω be an isoperimetric estimator of (Ω, d, μ) . Let $G \subset \Omega$ be an open set, and let $Z = Z([0, \mu(G)])$ be a r.i. space on $[0, \mu(G)]$. Then*

$$R(t) := \left\| \frac{1}{\min(I_\Omega(s), I_\Omega(\mu(G) - s))} \chi_{(0,t)}(s) \right\|_Z \leq 2 \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_Z, \quad 0 < t < \mu(G).$$

PROOF. If $0 < t < \mu(G)/2$, then, for $0 < s < t$, $\min(I_\Omega(s), I_\Omega(\mu(G) - s)) = I_\Omega(s)$. Consequently,

$$R(t) = \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_Z.$$

If $\mu(G)/2 < t < \mu(G)$, then

$$\begin{aligned} R(t) &\leq \left\| \frac{1}{\min(I_\Omega(s), I_\Omega(\mu(G) - s))} \chi_{(0, \mu(G)/2)}(s) \right\|_Z + \left\| \frac{1}{\min(I_\Omega(s), I_\Omega(\mu(G) - s))} \chi_{(\mu(G)/2, t)}(s) \right\|_Z \\ &= \left\| \frac{1}{I_\Omega(s)} \chi_{(0, \mu(G)/2)}(s) \right\|_Z + \left\| \frac{1}{I_\Omega(\mu(G) - s)} \chi_{(\mu(G)/2, t)}(s) \right\|_Z \\ &\leq \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_Z + \left\| \frac{1}{I_\Omega(\mu(G) - s)} \chi_{(\mu(G)/2, t)}(s) \right\|_Z. \end{aligned}$$

To estimate the second term on the right hand side, notice that, by the properties of $I_\Omega(s)$, the functions

$$\frac{1}{I_\Omega(s)} \chi_{(t - \mu(G)/2, \mu(G)/2)}(s) \text{ and } \frac{1}{I_\Omega(\mu(G)/2 - s)} \chi_{(\mu(G)/2, t)}(s)$$

are equimeasurable (with respect to the Lebesgue measure), thus

$$\begin{aligned} \left\| \frac{1}{I_\Omega(\mu(G)/2 - s)} \chi_{(\mu(G)/2, t)}(s) \right\|_Z &= \left\| \frac{1}{I_\Omega(s)} \chi_{(t - \mu(G)/2, \mu(G)/2)}(s) \right\|_Z \\ &\leq \left\| \frac{1}{I_\Omega(s)} \chi_{(0, \mu(G)/2)}(s) \right\|_{\bar{X}'} \\ &\leq \left\| \frac{1}{I_\Omega(s)} \chi_{(0, t)}(s) \right\|_Z. \end{aligned}$$

Consequently,

$$R(t) \leq 2 \left\| \frac{1}{I_\Omega(s)} \chi_{(0, t)}(s) \right\|_Z.$$

□

LEMMA 4. *Let (Ω, d, μ) be a metric measure space and let X be a r.i. space on Ω . Let $G \subset \Omega$ be an open connected subset such that an isoperimetric inequality relative to G holds true. Assume that the isoperimetric profile I_Ω of (Ω, d, μ) satisfies*

$$\left\| \frac{1}{I_\Omega(s)} \right\|_{\bar{X}'} < \infty.$$

Suppose that $f \in X + S_X$ is such that

$$\int_0^{\mu(\Omega)} \frac{K \left(\phi_X(t) \left\| \frac{1}{I_\Omega(s)} \chi_{(0, t)}(s) \right\|_{\bar{X}'}, f; X, S_X \right)}{\phi_X(t)} \frac{dt}{t} < \infty.$$

Then, for μ -almost every $x, y \in G$, the following inequality holds:

$$|f(x) - f(y)| \leq 2 \max(1, \frac{2}{C_G}) \int_0^{\mu(G)} \frac{K \left(\phi_X(t) \left\| \frac{1}{I_\Omega(s)} \chi_{(0, t)}(s) \right\|_{\bar{X}'}, f; X, S_X \right)}{\phi_X(t)} \frac{dt}{t}.$$

PROOF. We shall use the following notation: $X = X(\Omega)$, $S_X = S_X(\Omega)$, $X_r = X_r(G)$, $S_{X_r} = S_{X_r}(G)$.

Given $f \in X + S_X$, then $f\chi_G \in X_r + S_{X_r}$. Since an isoperimetric inequality relative to G holds true, we can apply Theorem 12 to obtain

$$(f\chi_G)_\mu^{**}(t) - (f\chi_G)_\mu^*(t) \leq 2 \frac{K(\psi_G(t), f\chi_G; X_r, S_{X_r})}{\phi_{X_r}(t)}, 0 < t < \mu(G),$$

where

$$\psi_G(t) = \frac{\phi_{X_r}(t)}{t} \left\| \frac{s}{J_G(s)} \chi_{(0, t)}(s) \right\|_{\bar{X}_r'}.$$

By [(2.8), Chapter 2],

$$\frac{K(\psi_G(t), f\chi_G; X_r, S_{X_r})}{\phi_{X_r}(t)} \leq \frac{K(\psi_G(t), f; X, S_X)}{\phi_X(t)}, 0 < t < \mu(G).$$

On the other hand,

$$\begin{aligned}
\psi_G(t) &= \frac{\phi_{X_r}(t)}{t} \left\| \frac{s}{J_G(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'_r} \\
&\leq \phi_{X_r}(t) \left\| \frac{1}{J_G(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'_r} \\
&= \phi_X(t) \left\| \frac{1}{J_G(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'} \\
&= \frac{\phi_X(t)}{C_G} \left\| \frac{1}{\min(I_\Omega(s), I_\Omega(\mu(G) - s))} \chi_{(0,t)}(s) \right\|_{\bar{X}'} \\
&\leq \frac{2\phi_X(t)}{C_G} \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'} \quad (\text{by Proposition 2}).
\end{aligned}$$

Summarizing, for $0 < t < \mu(G)$,

$$(f\chi_G)_\mu^{**}(t) - (f\chi_G)_\mu^*(t) \leq 2 \frac{K \left(\frac{2}{C_G} \phi_X(t) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f; X, S_X \right)}{\phi_X(t)}.$$

Letting $r = \frac{\mu(G)}{\mu(\Omega)}$, and $t = zr < \mu(G)$, we have

$$(f\chi_G)_\mu^{**}(zr) - (f\chi_G)_\mu^*(zr) \leq 2 \frac{K \left(\frac{2}{C_G} \phi_X(zr) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,zr)}(s) \right\|_{\bar{X}'}, f; X, S_X \right)}{\phi_X(zr)}.$$

Integrating the previous inequality we obtain

$$\begin{aligned}
&\int_0^{\mu(\Omega)} \left[(f\chi_G)_\mu^{**}(zr) - (f\chi_G)_\mu^*(zr) \right] \frac{dz}{z} \\
&\leq 2 \int_0^{\mu(\Omega)} \frac{K \left(\frac{2}{C_G} \phi_X(zr) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,zr)}(s) \right\|_{\bar{X}'}, f; X, S_X \right)}{\phi_X(zr)} \frac{dz}{z} \\
&= 2 \int_0^{\mu(G)} \frac{K \left(\frac{2}{C_G} \phi_X(z) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,z)}(s) \right\|_{\bar{X}'}, f; X, S_X \right)}{\phi_X(z)} \frac{dz}{z} \\
&\leq 2 \max(1, \frac{2}{C_G}) \int_0^{\mu(G)} \frac{K \left(\phi_X(z) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,z)}(s) \right\|_{\bar{X}'}, f; X, S_X \right)}{\phi_X(z)} \frac{dz}{z}.
\end{aligned}$$

Using the formula

$$\frac{d}{dz} \left\{ - (f\chi_G)_\mu^{**}(zr) \right\} = \frac{(f\chi_G)_\mu^{**}(zr) - (f\chi_G)_\mu^*(zr)}{z},$$

we get

$$\begin{aligned}
&\text{ess sup}(f\chi_G) - \frac{1}{\mu(G)} \int_0^{\mu(G)} (f\chi_G)_\mu^*(s) ds \\
&= (f\chi_G)_\mu^{**}(0) - (f\chi_G)_\mu^{**}(\mu(\Omega)r) \\
&\leq 2 \max(1, \frac{2}{C_G}) \int_0^{\mu(G)} \frac{K \left(\phi_X(z) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,z)}(s) \right\|_{\bar{X}'}, f; X, S_X \right)}{\phi_X(z)} \frac{dz}{z}.
\end{aligned}$$

Similarly, considering $-f\chi_G$, instead of $f\chi_G$, we obtain

$$\begin{aligned} & \frac{1}{\mu(G)} \int_0^{\mu(G)} (-f\chi_G)_\mu^*(s) ds - \text{ess inf}(f\chi_G) \\ & \leq 2 \max(1, \frac{2}{C_G}) \int_0^{\mu(G)} \frac{K\left(\phi_X(z) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,z)}(s) \right\|_{\bar{X}'}, f; X, S_X\right)}{\phi_X(z)} \frac{dz}{z}. \end{aligned}$$

Since $f\chi_G$ and $-f\chi_G$ are both supported on G , we have

$$\int_0^{\mu(G)} (f\chi_G)_\mu^*(s) ds = \int_G f d\mu \quad \text{and} \quad \int_0^{\mu(G)} (-f\chi_G)_\mu^*(s) ds = - \int_G f d\mu.$$

Adding these results, we have that for μ -almost every $x, y \in G$,

$$\begin{aligned} |f(x) - f(y)| & \leq \text{ess sup}(f\chi_G) - \text{ess inf}(f\chi_G) \\ & \leq 4 \max(1, \frac{2}{C_G}) \int_0^{\mu(G)} \frac{K\left(\phi_X(z) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,z)}(s) \right\|_{\bar{X}'}, f; X, S_X\right)}{\phi_X(z)} \frac{dz}{z}. \end{aligned}$$

□

THEOREM 13. *Let (Ω, d, μ) be a metric space with the relative isoperimetric property. Let X be a r.i. space on Ω such that*

$$\left\| \frac{1}{I_\Omega(s)} \right\|_{\bar{X}'} < \infty.$$

Suppose that $f \in X + S_X$ satisfies

$$\int_0^{\mu(\Omega)} \frac{K\left(\phi_X(t) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f; X, S_X\right)}{\phi_X(t)} \frac{dt}{t} < \infty,$$

then f is essentially bounded and essentially continuous.

PROOF. The essential boundedness follows from Lemma 3. The essential continuity of f is consequence of previous Lemma and the relative isoperimetric property. Indeed, given $\varepsilon > 0$ we can pick $\eta > 0$ such that

$$\int_0^\eta \frac{K\left(\phi_X(t) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f; X, S_X\right)}{\phi_X(t)} \frac{dt}{t} < \varepsilon.$$

Given $x \in \Omega$, by the relative isoperimetric property, we know that there exist $\delta(x)$ and a constant $C > 0$, such that, for any ball $B_\alpha(x)$ centered on x with measure $\mu(B_\alpha(x)) = \alpha$ ($0 < \alpha < \delta(x)$), we have

$$I_{B_\alpha(x)}(s) \geq C \min(I_\Omega(s), I_\Omega(\alpha - s)), \quad 0 < s < \alpha.$$

Pick $B_\alpha(x)$ with $\alpha < \eta$, then Lemma 4 ensures that for μ -almost every $z, y \in B_\alpha(x)$

$$\begin{aligned}
|f(z) - f(y)| &\leq 4 \max(1, \frac{2}{C}) \int_0^\alpha \frac{K\left(\phi_X(t) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f; X, S_X\right)}{\phi_X(t)} \frac{dt}{t} \\
&\leq 4 \max(1, \frac{2}{C}) \int_0^\eta \frac{K\left(\phi_X(t) \left\| \frac{1}{I_\Omega(s)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f; X, S_X\right)}{\phi_X(t)} \frac{dt}{t} \\
&\leq 4 \max(1, \frac{2}{C}) \varepsilon.
\end{aligned}$$

□

CHAPTER 5

Examples and Applications

1. Summary

We verify the relative isoperimetric property for a number of concrete examples. As a consequence we shall show in detail how our methods provide a unified treatment of embeddings of Sobolev and Besov spaces into spaces of continuous functions in different contexts.

2. Euclidean domains

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (i.e. a bounded, open and connected set). For a measurable function $u : \Omega \rightarrow \mathbb{R}$, let

$$u^+ = \max(u, 0) \text{ and } u^- = \min(u, 0).$$

Let $X = X(\Omega)$ be a r.i. space on Ω . The Sobolev space $W_X^1(\Omega) := W_X^1$ is the space of functions $f \in X$ of those real-valued weakly differentiable functions on Ω such whose first-order derivatives belong to X .

In this setting the basic rearrangement inequality holds for all $f \in W_{L^1}^1$

$$(2.1) \quad |f|^{**}(t) - |f|^*(t) \leq \frac{t}{I_\Omega(t)} \frac{1}{t} \int_0^t |\nabla f|^*(s) ds, 0 < t < |\Omega|,$$

holds (rearrangements are taken with respect to the Lebesgue measure). We indicate briefly the proof using the method of [69].

It is well known (see for example [5], [96, Theorem 2.1.4]) that if $u \in W_{L^1}^1$ ($= W_1^1$) then $u^+, u^- \in W_1^1$ and

$$\nabla u^+ = \nabla u \chi_{\{u>0\}} \text{ and } \nabla u^- = \nabla u \chi_{\{u<0\}}.$$

For given a measurable function g and $0 < t_1 < t_2$, the truncation $g_{t_1}^{t_2}$ of g is defined by

$$g_{t_1}^{t_2} = \min\{\max\{0, f - t_1\}, t_2 - t_1\}.$$

It follows that if $g \in W_1^1$, then $g_{t_1}^{t_2} \in W_1^1$ and, in fact,

$$\nabla g_{t_1}^{t_2} = \nabla g \chi_{\{t_1 < g < t_2\}}.$$

In other words, W_1^1 is invariant by truncation. On the other hand, given $g \in W_1^1$, the Federer-Fleming-Rishel co-area formula (cf. [37]) states that

$$\int_\Omega |\nabla g(x)| dx = \int_{-\infty}^{\infty} P_\Omega(f > s) ds.$$

Applying this result to $|g|_{t_1}^{t_2}$, we get

$$\begin{aligned}
 (2.2) \quad \int_{\{t_1 < |g| < t_2\}} |\nabla |g|(x)| dx &= \int_0^\infty P_\Omega(|g|_{t_1}^{t_2} > s) ds \\
 &\geq \int_0^\infty I_\Omega(\mu_{|g|_{t_1}^{t_2}}(s)) ds \quad (\text{isoperimetric inequality}) \\
 &= \int_0^{t_2-t_1} I_\Omega(\mu_{|g|_{t_1}^{t_2}}(s)) ds.
 \end{aligned}$$

Observe that, for $0 < s < t_2 - t_1$,

$$|\{|f| \geq t_2\}| \leq \mu_{|f|_{t_1}^{t_2}}(s) \leq |\{|f| > t_1\}|.$$

Consequently, by the properties of I_Ω , we have

$$\int_0^{t_2-t_1} I(\mu_{|g|_{t_1}^{t_2}}(s)) ds \geq (t_2 - t_1) \min(I_\Omega(|\{|g| \geq t_2\}|), I_\Omega(|\{|g| \geq t_1\}|)).$$

For $s > 0$ and $h > 0$, pick $t_1 = |f|^*(s+h)$, $t_2 = |f|^*(s)$, then

$$(2.3) \quad s \leq |\{|g| \geq |g|^*(s)\}| \leq \mu_{|g|_{t_1}^{t_2}}(s) \leq |\{|g(x)| > |g|^*(s+h)\}| \leq s+h.$$

Combining (2.2) and (2.3) we have,

$$(|g|^*(s) - |g|^*(s+h)) \min(I_\Omega(s+h), I_\Omega(s)) \leq \int_{\{|g|^*(s+h) < |g| < |g|^*(s)\}} |\nabla |g|(x)| dx.$$

At this stage we can continue as in [69], and we obtain that if $f \in W_1^1$, then (2.1) holds. Moreover, $|f|^*$ is locally absolutely continuous, and

$$(2.4) \quad \int_0^t |(-|f|^*)'(\cdot) I_\Omega(\cdot)|^*(s) ds \leq \int_0^t |\nabla |f|^*(s)| ds.$$

From here, using the same approximation method provided in the proof of Theorem 12 Chapter 4, we find that, if $f \in W_1^1$, then for $0 < t < |\Omega|$,

$$(2.5) \quad f^{**}(t) - f^*(t) \leq \frac{t}{I_\Omega(t)} \frac{1}{t} \int_0^t |\nabla |f|^*(s)| ds.$$

Indeed, first assume that f is bounded from below, and let $c = \inf_\Omega f$, then, since $f - c \geq 0$, we can apply (2.1) to $f - c$, and we obtain

$$|f - c|^{**}(t) - |f - c|^*(t) \leq \frac{t}{I_\Omega(t)} \frac{1}{t} \int_0^t |\nabla (f - c)|^*(s) ds.$$

Since $|f - c|^*(t) = (f - c)^*(t)$ and $f^{**}(t) - f^*(t) = (f - c)^{**}(t) - (f - c)^*(t)$, we get

$$f^{**}(t) - f^*(t) \leq \frac{t}{I_\Omega(t)} \frac{1}{t} \int_0^t |\nabla |f|^*(s)| ds.$$

If f is not bounded from below, let $f_n = \max(f, -n)$, $n = 1, 2, \dots$. The previous discussion gives

$$\begin{aligned}
 (f_n)^{**}(t) - (f_n)^*(t) &\leq \frac{t}{I_\Omega(t)} \frac{1}{t} \int_0^t |\nabla |f_n|^*(s)| ds \\
 &\leq \frac{t}{I_\Omega(t)} \frac{1}{t} \int_0^t |\nabla |f|^*(s)| ds.
 \end{aligned}$$

We now take limits. To compute the left hand side we observe that $f_n(x) \rightarrow f(x)$ μ -a.e., and $|f_n| \leq |f|$, implies (cf. [13]) $(f_n)^{**}(t) - (f_n)^*(t) \xrightarrow{n \rightarrow \infty} f^{**}(t) - f^*(t)$, ($0 < t < \mu(\Omega)$).

Let $X = X(\Omega)$ be a r.i. space on Ω . The homogeneous Sobolev space \dot{W}_X^1 is defined by means of the quasi norm

$$\|u\|_{\dot{W}_X^1} := \|\nabla u\|_X.$$

We consider the corresponding K -functional

$$K(t, f; X, \dot{W}_X^1) = \inf\{\|f - g\|_X + t\|g\|_{\dot{W}_X^1}\}.$$

The previous discussion shows that all the results of Chapters 3 and 4 remain valid for functions in \dot{W}_X^1 or $X + \dot{W}_X^1$.

2.1. Sobolev spaces defined on Lipschitz domains of \mathbb{R}^n . We now discuss assumptions on the domain that translate into good estimates for the corresponding isoperimetric profiles.

In this section we consider Sobolev spaces defined on Lipschitz domains of \mathbb{R}^n .

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then the isoperimetric profile, satisfies (see for example [74])

$$(2.6) \quad I_\Omega(t) = c(n) \min(t, |\Omega| - t)^{\frac{n-1}{n}}.$$

For any open ball that $B_\alpha \subset \Omega$ with $|B_\alpha| = \alpha$, we know that inequality (see for example [74] or [96])

$$I_{B_\alpha}(t) \geq q(n) \min(t, \alpha - t)^{\frac{n-1}{n}}, \quad 0 < t < \alpha,$$

holds, where $q(n)$ is a constant that only depends on n . Since

$$c(n) \min(t, \alpha - t)^{\frac{n-1}{n}} = \min(I_\Omega(t), I_\Omega(\alpha - t)) \quad 0 < t < \alpha,$$

we get that there is a constant $C = C(n)$ that just depends on n such that

$$I_{B_\alpha}(t) \geq C \min(I_\Omega(t), I_\Omega(\alpha - t)) \quad 0 < t < \alpha,$$

i.e. the metric space $(\Omega, |\cdot|, dm)$ has the relative isoperimetric property.

THEOREM 14. *Let $X = X(\Omega)$ be a r.i. space on Ω , then*

$$\dot{W}_X^1(\Omega) \subset L^\infty \Leftrightarrow \left\| t^{1/n-1} \chi_{(0, |\Omega|/2)} \right\|_{X'} < \infty \Leftrightarrow \dot{W}_X^1(\Omega) \subset C_b(\Omega).$$

($C_b(\Omega)$ denotes the space of real valued continuous bounded functions defined on Ω).

PROOF. Let $f \in \dot{W}_X^1(\Omega)$. We have shown in the previous section that $|f|^*$ is locally absolutely continuous (cf. [60] and [69]) we can write

$$\begin{aligned}
 (2.7) \quad \|f\|_{L^\infty} - |f|^*(|\Omega|) &= |f|^*(0) - |f|^*(|\Omega|) = \int_0^{|\Omega|} (-|f^*|)'(s) ds \\
 &= \int_0^{|\Omega|} (-|f^*|)'(s) I_\Omega(s) \frac{ds}{I_\Omega(s)} \\
 &\leq \|(-|f^*|)'(s) I_\Omega(s)\|_{\bar{X}} \left\| \frac{1}{I_\Omega(t)} \right\|_{\bar{X}'} \quad (\text{by Hölder's inequality}) \\
 &\leq \|\nabla f\|_X \left\| \frac{1}{I_\Omega(t)} \right\|_{\bar{X}'} \quad (\text{by (2.4)}).
 \end{aligned}$$

Conversely, if $\dot{W}_X^1(\Omega) \subset L^\infty$ then

$$\left\| f - \int_\Omega f \right\|_{L^\infty} \leq c \|\nabla f\|_X.$$

Since Ω has bounded Lipschitz boundary, this is equivalent (cf. [69] and [67, Theorem 2]) to the existence of an absolute constant $C > 0$ such that for all $g \in \bar{X}$, $g \geq 0$

$$\left\| \int_t^{|\Omega|/2} \frac{g(s)}{I_\Omega(s)} ds \right\|_{L^\infty} \leq C \|g\|_{\bar{X}}.$$

Thus

$$\sup_{\|g\|_{\bar{X}} \leq 1} \int_0^{|\Omega|/2} \frac{|g|(s)}{I_\Omega(s)} ds = \sup_{\|g\|_{\bar{X}} \leq 1} \int_0^{|\Omega|} |g|(s) \frac{\chi_{(0,1|\Omega|/2)}(s)}{I_\Omega(s)} ds < C,$$

which, by duality, implies that

$$\left\| \frac{1}{I_\Omega(s)} \chi_{(0,|\Omega|/2)} \right\|_{\bar{X}'} < \infty.$$

Now, since I_Ω is symmetric about the point $|\Omega|/2$, it follows that

$$\begin{aligned}
 \left\| \frac{1}{I_\Omega(s)} \right\|_{\bar{X}'} &\leq \left\| \frac{1}{I_\Omega(s)} \chi_{(0,|\Omega|/2)} \right\|_{\bar{X}'} + \left\| \frac{1}{I_\Omega(s)} \chi_{(|\Omega|/2,|\Omega|)} \right\|_{\bar{X}'} \\
 &= 2 \left\| \frac{1}{I_\Omega(s)} \chi_{(0,|\Omega|/2)} \right\|_{\bar{X}'} < \infty.
 \end{aligned}$$

We will now show that

$$\left\| \frac{1}{I_\Omega(s)} \chi_{(0,|\Omega|/2)} \right\|_{\bar{X}'} \simeq \left\| t^{1/n-1} \chi_{(0,|\Omega|/2)} \right\|_{\bar{X}'} < \infty \Rightarrow \dot{W}_X^1(\Omega) \subset C_b(\Omega).$$

Let $f \in \dot{W}_X^1(\Omega)$, $x \in \Omega$. Consider any open ball B_α such that $x \in B_\alpha \subset \Omega$ with $|B_\alpha| = \alpha$. An easy computation shows that

$$\left\| \frac{1}{\min(t, \alpha - t)^{\frac{n-1}{n}}} \right\|_{\bar{X}'} \simeq \left\| t^{1/n-1} \chi_{(0,\alpha/2)} \right\|_{\bar{X}'}.$$

Applying the inequality (2.5) to $f\chi_{B_\alpha}$ and integrating, we get

$$\begin{aligned} \operatorname{ess\,sup}(f\chi_{B_\alpha}) - \frac{1}{\alpha} \int f\chi_{B_\alpha}(x)dx &= \int_0^\alpha ((f\chi_{B_\alpha})^{**}(t) - (f\chi_{B_\alpha})^*(t)) \frac{dt}{t} \\ &\leq \int_0^\alpha \left(\frac{t}{I_{B_\alpha}(t)} \frac{1}{t} \int_0^t |\nabla f\chi_{B_\alpha}|^*(s)ds \right) \frac{dt}{t} \\ &\leq \left\| \frac{1}{t} \int_0^t |\nabla f\chi_{B_\alpha}|^*(s)ds \right\|_{\bar{X}} \left\| \frac{1}{I_{B_\alpha}(t)} \right\|_{\bar{X}'} \\ &\leq c(n, X) \|\nabla f\chi_{B_\alpha}\|_X \left\| t^{1/n-1} \chi_{(0, \alpha/2)} \right\|_{\bar{X}'}. \end{aligned}$$

Similarly, considering $-f\chi_{B_\alpha}$, we get

$$-\operatorname{ess\,sin}(f\chi_{B_\alpha}) + \frac{1}{\alpha} \int f\chi_{B_\alpha}(x)dx \leq c(n, X) \|\nabla f\chi_{B_\alpha}\|_X \left\| t^{1/n-1} \chi_{(0, \alpha/2)} \right\|_{\bar{X}'}.$$

Adding these inequalities we see that

$$\operatorname{ess\,sup}(f\chi_{B_\alpha}) - \operatorname{ess\,sin}(f\chi_{B_\alpha}) \leq c \|\nabla f\chi_{B_\alpha}\|_X \left\| t^{1/n-1} \chi_{(0, \alpha/2)} \right\|_{\bar{X}'}.$$

Thus, for almost every $y \in B_\alpha$,

$$|f(x) - f(y)| \leq c(n) \|\nabla f\chi_{B_\alpha}\|_X \left\| t^{1/n-1} \chi_{(0, \alpha/2)} \right\|_{\bar{X}'}.$$

□

REMARK 5. Let us consider the case when $X = L^p$, with $p > n$. An elementary computation shows that

$$\left\| t^{1/n-1} \chi_{(0, \alpha/2)} \right\|_{p'} \leq c_{(n,p)} \alpha^{\frac{1}{n}(1-\frac{n}{p})}.$$

Since $|B_\alpha| = \alpha$, then $\alpha^{1/n}$ is c_n times the radius of the ball, thus, for almost every $y, z \in B_\alpha$ such that $|y - z| = c_n \alpha^{1/n}$, we get

$$|f(y) - f(z)| \leq c(n, p) |y - z|^{(1-\frac{n}{p})} \|\nabla f\|_p.$$

The result fails if $p = n$. However, if we consider the smaller Lorentz space $X = L^{n,1} \subset L^n$, then $X' = L^{\frac{n}{n-1}, \infty}$, and

$$\left\| t^{1/n-1} \chi_{(0, \alpha/2)} \right\|_{L^{\frac{n}{n-1}, \infty}} = \sup_{0 < s < \alpha/2} s^{\frac{1}{n}-1} s^{1-\frac{1}{n}} = 1.$$

Thus, for almost every $y, z \in B_\alpha$ such that $|y - z| = \alpha^{1/n}$, we have that

$$|f(y) - f(z)| \leq c(n) \|\nabla f\chi_{B_\alpha}\|_{L^{n,1}} = c(n) \int_0^{|y-z|^n} s^{1/n} |\nabla f|^*(s) \frac{ds}{s}.$$

The essential continuity of f follows. Thus we recover the classical result independently due to Stein [88] and C. P. Calderón [23].

REMARK 6. See [25] for a related result, using a different method and involving Orlicz norms.

2.2. Spaces defined in terms of the modulus of continuity on Lipschitz domains of \mathbb{R}^n . For Euclidean domains Ω with Lipschitz boundary it is known that (cf. [54, Theorem 1], [13, Chapter 5, exercise 13, pag. 430]),

$$K(t, g; X(\Omega), \dot{W}_X^1(\Omega)) \simeq \omega_X(g, t), \quad 0 < t < |\Omega|,$$

where

$$\omega_X(f, t) = \sup_{0 < |h| \leq t} \|(f(\cdot + h) - f(\cdot))\chi_{\Omega(h)}\|_{L^p(\Omega)},$$

with $\Omega(h) = \{x \in \Omega : x + \rho h \in \Omega, \quad 0 \leq \rho \leq 1\}$ and $h \in \mathbb{R}^n$.

Moreover, as we have seen, $(\Omega, |\cdot|, dm)$ has the relative isoperimetric property. Consequently, by Theorem 13, we have

THEOREM 15. *Let X be a r.i. space on Ω , such that*

$$(2.8) \quad \left\| \frac{1}{\min(t, |\Omega| - t)^{\frac{n-1}{n}}} \right\|_{\bar{X}'} < \infty.$$

If $f \in X + \dot{W}_X^1$ satisfies

$$\int_0^1 \frac{\omega_X \left(f, \phi_X(t) \left\| \frac{1}{\min(t, |\Omega| - t)^{\frac{n-1}{n}}} \chi_{(0,t)}(s) \right\|_{\bar{X}'} \right)}{\phi_X(t)} \frac{dt}{t} < \infty,$$

then, f is essentially bounded and essentially continuous.

In particular, when $X = L^p$, we obtain

THEOREM 16. *If $n/p < 1$, then there exists a constant $c > 0$, such that*

$$|f(y) - f(z)| \leq C \int_0^{|y-z|} \frac{\omega_p(f, t) dt}{t^{n/p} t}.$$

PROOF. Since $p > n$ an elementary computation shows that condition (2.8) holds. In fact,

$$\left\| \frac{1}{I_{B_\alpha}(s)} \chi_{(0,t)}(s) \right\|_{L^{p'}(B_\alpha)} \leq c_{(n,p)} t^{\frac{1}{n} - \frac{1}{p}}.$$

Thus, Lemma 4 ensures that for some absolute constant $D_{(n,p)} > 0$, and for almost every $y, z \in B_\alpha$,

$$\begin{aligned} |f(y) - f(z)| &\leq D_{(n,p)} \int_0^\alpha \frac{K \left(t^{1/n}, f; L^p(\Omega), \dot{W}_p^1(\Omega) \right)}{t^{1/p}} \frac{dt}{t} \\ &= D_{(n,p)} \int_0^{\alpha^{1/n}} \frac{K \left(t, f; L^p(\Omega), \dot{W}_p^1(\Omega) \right)}{t^{n/p}} \frac{dt}{t}. \end{aligned}$$

Since $|B_\alpha| = \alpha$, $\alpha^{1/n}$ is a constant times the radius of the ball, and therefore for almost every $y, z \in B_\alpha$ such that $|y - z| = c\alpha^{1/n}$

$$\begin{aligned} |f(y) - f(z)| &\leq \int_0^{|y-z|} \frac{K \left(t, f; L^p(\Omega), \dot{W}_p^1(\Omega) \right)}{t^{n/p}} \frac{dt}{t} \\ &\simeq \int_0^{|y-z|} \frac{\omega_p(f, t) dt}{t^{n/p} t}. \end{aligned}$$

The essential continuity of f follows. □

REMARK 7. In Chapter 10 we shall discuss the connection with A. Garsia's work.

THEOREM 17. Let X be a r.i. space such that $\overline{\alpha}_{\Lambda(X')} < \frac{1}{n}$. If f satisfies

$$\int_0^{|\Omega|} \frac{\omega_X(f, t)}{\phi_X(t^{1/n})} \frac{dt}{t} < \infty,$$

then, f is essentially continuous.

PROOF. It is enough to prove

$$R(t) = \left\| \frac{1}{\min(s, (|\Omega| - s))^{\frac{n-1}{n}}} \chi_{(0,t)}(s) \right\|_{\bar{X}'} \leq c_{(n,X)} t^{\frac{1}{n}} \phi_{\bar{X}}(t), \quad 0 < t < |\Omega|.$$

Recall that if $\underline{\alpha}_{\Lambda(X')} > 0$, the fundamental function of X' satisfies (see [85, Theorem 2.4])

$$d\phi_{X'}(s) \simeq \frac{\phi_{X'}(s)}{s},$$

and, moreover, for every $0 < \gamma < \underline{\alpha}_{\Lambda(X')}$ the function $\phi_{X'}(s)/s^\gamma$ is almost increasing (i.e. $\exists c > 0$ s.t. $\phi_{X'}(s)/s^\gamma \leq c\phi_{X'}(t)/t^\gamma$ whenever $t \geq s$). Pick $0 < \beta$ such that (notice that $\overline{\alpha}_{\Lambda(X)} < \frac{1}{n}$ implies that $\underline{\alpha}_{\Lambda(X')} > 1 - \frac{1}{n}$)

$$1 - \frac{1}{n} + \beta < \underline{\alpha}_{\Lambda(X')}.$$

Since I_Ω is symmetric about the point $1/2$, we get

$$R(t) \simeq \left\| \frac{1}{s^{\frac{n-1}{n}}} \chi_{(0,t)}(s) \right\|_{\bar{X}'},$$

and

$$\begin{aligned} R(t) &\leq \int_0^t s^{\frac{1}{n}-1} d\phi_{X'}(s) \\ &\simeq \int_0^t s^{\frac{1}{n}-1} \frac{\phi_{X'}(s)}{s} ds \\ &= \int_0^t \frac{\phi_{X'}(s)}{s^{1-\frac{1}{n}+\beta}} \frac{ds}{s^{1-\beta}} \\ &\preceq \frac{\phi_{X'}(t)}{t^{1-\frac{1}{n}+\beta}} \int_0^t \frac{ds}{s^{1-\beta}} \\ &\simeq \frac{\phi_{X'}(t)}{t^{1-\frac{1}{n}}} \\ &= t^{\frac{1}{n}} \phi_{\bar{X}}(t). \end{aligned}$$

□

3. Domains of Maz'ya's class \mathcal{J}_α ($1 - 1/n \leq \alpha < 1$)

DEFINITION 2. (See [73], [74]) A domain $\Omega \subset \mathbb{R}^n$ (with finite measure) belongs to the class \mathcal{J}_α ($1 - 1/n \leq \alpha < 1$) if there exists a constant $M \in (0, |\Omega|)$ such that

$$U_\alpha(M) = \sup \frac{|\mathcal{S}|^\alpha}{P_\Omega(\mathcal{S})} < \infty,$$

where the sup is taken over all \mathcal{S} open bounded subsets of Ω such that $\Omega \cap \partial\mathcal{S}$ is a manifold of class C^∞ and $|\mathcal{S}| \leq M$, (in which case we will say that \mathcal{S} is an admissible subset) and where for a measurable set $E \subset \Omega$, $P_\Omega(E)$ is the De Giorgi perimeter of E in Ω defined by

$$P_\Omega(E) = \sup \left\{ \int_E \operatorname{div} \varphi \, dx : \varphi \in [C_0^1(\Omega)]^n, \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

By an approximation process it follows that if Ω is a bounded domain in \mathcal{J}_α , then there exists a constant $c_\Omega > 0$ such that, for all measurable set $E \subset \Omega$ with $|E| \leq |\Omega|/2$, we have

$$P_\Omega(E) \geq c_\Omega |E|^\alpha.$$

Since E and $\Omega \setminus E$ have the same boundary measure, we obtain the following isoperimetric inequality

$$I_\Omega(t) \geq c_\Omega \min(t, |\Omega| - t)^\alpha := J_\Omega(t), \quad 0 < t < |\Omega|.$$

For example, if Ω is a bounded domain, star shaped with respect to a ball, or having the cone property, or is a Lipschitz domain, then belongs to the class $\mathcal{J}_{1-1/n}$; if Ω is a s -John domain then $\Omega \in \mathcal{J}_{(n-s)/n}$; if Ω is a domain with one β -cusp then it belongs to the Maz'ya class $\mathcal{J}_{\frac{\beta(n-1)}{\beta(n-1)+1}}$.

THEOREM 18. Let $\Omega \in \mathcal{J}_\alpha$, and let X a r.i. space on Ω . Suppose that

$$(3.1) \quad \left\| \frac{1}{J_\Omega(t)} \right\|_{X'} < \infty.$$

Then,

(1)

$$\dot{W}_X^1(\Omega) \subset C_b(\Omega).$$

(2) If $f \in X + \dot{W}_X^1$ satisfies

$$\int_0^{\mu(\Omega)} \frac{K\left(\phi_X(t) \left\| \frac{1}{J_\Omega(t)} \chi_{(0,t)}(s) \right\|_{X'}, f; X, \dot{W}_X^1\right) dt}{\phi_X(t)} \frac{dt}{t} < \infty,$$

then f is essentially bounded and essentially continuous.

PROOF. Part 1. The inclusion $\dot{W}_X^1(\Omega) \subset L^\infty$ follows in the same way as the corresponding part of Theorem 14 (cf. inequality (2.7)). To prove the essential continuity we proceed as follows. Let B be any ball contained on Ω with $|B| \leq \min(1, |\Omega|/2)$. Notice that if $f \in \dot{W}_X^1(\Omega)$, then $f\chi_B \in \dot{W}_X^1(B)$. Now, since B is a Lip domain, by Theorem 14, we just need to verify that

$$\left\| t^{1/n-1} \chi_{(0, |B|/2)} \right\|_{X'} < \infty.$$

Since $1 - 1/n \leq \alpha < 1$, and $0 < t < |B|/2 < 1$, we have

$$\sup_{0 < t < |B|/2} t^{\alpha+1/n-1} = \left(\frac{|B|}{2} \right)^{\alpha+1/n-1}.$$

Thus,

$$\begin{aligned} \left\| t^{1/n-1} \chi_{(0,|B|/2)} \right\|_{\bar{X}'} &= \left\| \frac{t^{\alpha+1/n-1}}{t^\alpha} \chi_{(0,|B|/2)} \right\|_{\bar{X}'} \\ &\leq \left(\frac{|B|}{2} \right)^{\alpha+1/n-1} \left\| \frac{1}{t^\alpha} \chi_{(0,|B|/2)} \right\|_{\bar{X}'} < \infty \quad (\text{by (3.1)}). \end{aligned}$$

Part 2. Let B be any ball contained on Ω with $|B| \leq \min(1, |\Omega|/2)$. Then

$$K \left(t, f\chi_B; X(B), \dot{W}_X^1(B) \right) \leq K \left(t, f; X, \dot{W}_X^1 \right).$$

Using the same argument given in the first part of the proof, we obtain

$$\begin{aligned} \left\| \frac{\chi_{(0,t)}(s)}{\min(t, |B| - t)^{\frac{n-1}{n}}} \right\|_{\bar{X}'} &\leq 2 \left\| \frac{\chi_{(0,t)}(s)}{t^{\frac{n-1}{n}}} \right\|_{\bar{X}'} \\ &\leq \left\| \frac{1}{J_\Omega(t)} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, \quad 0 < t < |B|. \end{aligned}$$

Therefore,

$$\int_0^{|B|} \frac{K \left(\phi_X(t) \left\| \frac{\chi_{(0,t)}(s)}{\min(t, |B| - t)^{\frac{n-1}{n}}} \right\|_{\bar{X}'}, f\chi_B; X(B), \dot{W}_X^1(B) \right)}{\phi_X(t)} \frac{dt}{t} < \infty,$$

and Theorem 15 applies. \square

4. Ahlfors Regular Metric Measure Spaces

In this section we consider Ahlfors regular spaces. In other words we assume that (Ω, d, μ) is a complete connected metric space and μ , is a Borel measure satisfying

$$(4.1) \quad c_\Omega r^k \leq \mu(B(x, r)) \leq C_\Omega r^k, \quad \forall x \in \Omega, \quad r \in (0, \text{diam}(\Omega)),$$

and, moreover, we assume the existence of constants $C > 0$ and $\lambda \geq 1$ such that the following (weak $(1, 1)$ –Poincaré) holds

$$(4.2) \quad \int_{B(x, r)} |u(y) - u_{B(x, r)}| d\mu(y) \leq Cr \int_{B(x, \lambda r)} |\nabla u(y)| d\mu(y),$$

whenever $u \in Lip(\Omega)$, where $u_{B(x, r)}$ denotes the mean value of u in B , i.e. $u_{B(x, r)} = \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u(s) d\mu(s)$.

Examples of spaces supporting a (weak) $(1, 1)$ –Poincaré inequality include Riemannian manifolds with lower bounds on the Ricci curvature, Carnot–Carathéodory groups, and more generally (in the case of doubling spaces) Carnot–Carathéodory spaces associated to smooth (or locally Lipschitz) vector fields satisfying Hörmander’s condition (see for example [4], [49] and the references quoted therein).

By a well known result of Hajlasz and Koskela (cf. [49]), (4.1) and (4.2) imply

$$\left(\int_{B(x, r)} |u(y) - u_{B(x, r)}|^{k/(k-1)} d\mu(y) \right)^{(k-1)/k} \leq D \int_{B(x, 2\lambda r)} |\nabla u(y)| d\mu(y)$$

with $C = (2C)^{(k-1)/k}$.

According [78] (see also the references quoted therein), given a Borel set $E \subset \Omega$, and $A \subset \Omega$ open, the relative perimeter of E in A , denoted by $P(E, A)$, is defined by

$$P(E, A) = \inf \left\{ \liminf_{h \rightarrow \infty} \int_A |\nabla u_h| d\mu : u_h \in Lip_{loc}(A), u_h \rightarrow \chi_E \text{ in } L^1_{loc}(A) \right\}.$$

LEMMA 5. *The following relative isoperimetric inequality holds:*

$$(4.3) \quad \min(\mu(E \cap B(x, r)), \mu(E^c \cap B(x, r))) \leq DP(E, B(x, r))^{k/(k-1)}.$$

PROOF. The proof of this result is contained in the proof of [4, Theorem 4.3], we include the details for the sake of completeness.

We can assume that (Ω, d, μ) is a length space, this means that any pair of points $x, y \in \Omega$, can be connected by a rectifiable curve of length $d(x, y)$. Indeed, by [86] (see also the appendix of [24]), Ahlfors regularity and weak $(1, 1)$ -Poincaré inequality imply that (Ω, d, μ) is quasi-convex; i.e., there exists a constant M depending only on c_Ω , C_Ω and C such that any pair of points $x, y \in \Omega$ can be connected by a rectifiable curve of length at most $Md(x, y)$. Hence, since the statement of the theorem is bi-Lipschitz invariant, we can simply replace d by the geodesic metric

$$\tilde{d}(x, y) := \inf \left\{ \sum_{i=1}^{n-1} d(x_{i+1}, x_i) : x_1 = x, x_n = y \right\}$$

associated to d . Since any ball in a length space is a John domain, from Corollary 9.8 in [49] we obtain that the following Poincaré inequality holds: for all $u \in Lip(B(x, r))$

$$\left(\int_{B(x, r)} |u(y) - u_{B(x, r)}|^{k/(k-1)} d\mu(y) \right)^{(k-1)/k} \leq D \int_{B(x, r)} |\nabla u(y)| d\mu(y).$$

Taking into account the definition of $P(E, B(x, r))$, and noticing that by a truncation argument we need only to consider sequences u_h converging to χ_E in $L^1(B(x, r))$, we obtain

$$\left(\int_{B(x, r)} |\chi_E(y) - (\chi_E)_{B(x, r)}|^{k/(k-1)} d\mu(y) \right)^{(k-1)/k} \leq DP(E, B(x, r)),$$

and the relative isoperimetric inequality (4.3) follows. \square

THEOREM 19. *Let $k > 1$ be the exponent satisfying (4.1), and let $B := B(x, r)$ be a ball. The following statements are equivalent.*

- (1) *For every set of finite perimeter E in Ω ,*

$$c(k, C) (\min(\mu(E \cap B), \mu(E^c \cap B)))^{\frac{k}{k-1}} \leq P(E, B).$$

where the constant $c(k, C)$ does not depend on B .

- (2) *$\forall u \in Lip(\Omega)$, the function $|u\chi_B|_\mu^*$ is locally absolutely continuous and, for $0 < t < \mu(B)$,*

$$c(k, C) \int_0^t \left| \left((-|u\chi_B|_\mu^*)'(\cdot) (\min(\cdot, \mu(B) - \cdot))^{\frac{k-1}{k}} \right)^* (s) \right| ds \leq \int_0^t |\nabla u\chi_B|_\mu^*(s) ds.$$

(3) *Oscillation inequality:* $\forall u \in Lip(\Omega)$ and, for $0 < t < \mu(B)$

$$(|u\chi_B|_\mu^{**}(t) - |u\chi_B|_\mu^*(t)) \leq \frac{t}{c(k, C) (\min(t, \mu(B) - t))^{\frac{k-1}{k}}} |\nabla u\chi_B|_\mu^{**}(t).$$

PROOF. Consider the metric space $(B, d|_B, \mu|_B)$, then the Theorem is a particular case of Theorem 1 of [69]. \square

The local version of Theorem 12 is

THEOREM 20. *Let X be a r.i. space on Ω and let $B \subset \Omega$ be an open ball. Then, for each $f \in X + S_X$,*

$$(f\chi_B)_\mu^{**}(t) - (f\chi_B)_\mu^*(t) \leq 2 \frac{K(\psi(t), f\chi_B; X, S_X)}{\phi_X(t)}, \quad 0 < t < \mu(B),$$

where

$$\psi(t) = \frac{\phi_X(t)}{t} \left\| \frac{s}{c(k, C) (\min(s, \mu(B) - s))^{\frac{k-1}{k}}} \chi_{(0,t)}(s) \right\|_{\bar{X}'}.$$

PROOF. Let $f \in X + S_X$, then $f\chi_B \in X_r(B) + S_{X_r}(B)$, where B is the metric space $(B, d|_B, \mu|_B)$. By Lemma 5 we know that

$$c(k, C) (\min(\mu(E \cap B), \mu(E^c \cap B)))^{\frac{k}{k-1}} \leq P(E, B).$$

Thus, for any Borel set $E \subseteq B$,

$$c(k, C) (\min(\mu(E), \mu(B) - \mu(E)))^{\frac{k}{k-1}} \leq P_B(E).$$

Consequently, $J_B(t) = c(k, C) (\min(t, \mu(B) - t))^{\frac{k}{k-1}}$ ($0 < t < \mu(B)$) is an isoperimetric estimator of $(B, d|_B, \mu|_B)$, and now we finish the proof in the same way as in Theorem 12. \square

THEOREM 21. *Let $f \in X + S_X$ and let B be an open ball, if*

$$\int_0^{\mu(B)} \frac{K\left(\phi_X(t) \left\| (\min(s, \mu(B) - s))^{1-1/k} \chi_{(0,t)}(s) \right\|_{\bar{X}'}, f\chi_B; X, S_X\right) dt}{\phi_X(t)} \frac{1}{t} < \infty$$

then, $f\chi_B$ is essentially bounded and essentially continuous.

PROOF. By the proof of the previous Theorem we know that

$$J_B(t) = c(k, C) (\min(t, \mu(B) - t))^{\frac{k}{k-1}}$$

is an isoperimetric estimator of $(B, d|_B, \mu|_B)$. For any open ball $B(x, r) \subset B$, it follows from Lemma 5 that, for $0 < s < \mu(Q_{B(x,r)})$,

$$\begin{aligned} c(k, C) (\min(t, \mu(B(x, r)) - t))^{\frac{k}{k-1}} &\leq c(k, C) \min(J_B(t), J_B(\mu(Q_{B(x,r)}) - t)) \\ &\leq P_{B(x,r)}(s). \end{aligned}$$

Therefore $(B, d|_B, \mu|_B)$ has the relative isoperimetric property and Theorem 13 applies. \square

REMARK 8. *In the particular case $X = L^p$, we can thus use the same argument given in Theorem 16 to obtain that for $k/p < 1$, there exists an absolute constant such that*

$$|f(y) - f(z)| \leq \int_0^{|y-z|} \frac{K(t, f\chi_B; X, S_X)}{t^{k/p}} \frac{dt}{t}, \quad y, z \in B.$$

CHAPTER 6

Fractional Sobolev inequalities in Gaussian measures

1. Introduction and Summary

As another application of our theory, in this chapter we consider in detail fractional logarithmic Sobolev inequalities. We will deal not only with Gaussian measures but also with measures that interpolate between Gaussian and exponential.

In the context of classical Gaussian measures a typical result in this chapter is given by the following fractional logarithmic Sobolev inequality. Let $d\gamma_n$ be the Gaussian measure on \mathbb{R}^n , let $1 \leq q < \infty$, $\theta \in (0, 1)$; then, there exists an absolute constant $c > 0$, independent of the dimension, such that (cf Theorem 24 below)

$$(1.1) \quad \left\{ \int_0^{1/2} |f|_{\gamma_n}^* (t)^q \left(\log \frac{1}{t} \right)^{\frac{q\theta}{2}} dt \right\}^{1/q} \leq c \|f\|_{B_{L^q}^{\theta,q}(\gamma_n)},$$

(here $B_{L^q}^{\theta,q}(\gamma_n)$ is the Gaussian Besov space, see (3.1) below). Note that if $q = 2$, (1.1) interpolates between the embedding that follows from the classical logarithmic Sobolev inequality (which corresponds to the case $\theta = 1$) and the trivial embedding $L^2 \subset L^2$ (the case $\theta = 0$). For related inequalities using semigroups see [6] and also [39].

More generally, we will also prove fractional Sobolev inequalities for tensor products of measures that, on the real line are defined as follows. Let $\alpha \geq 0$, $r \in [1, 2]$ and $\gamma = \exp(2\alpha/(2-r))$, ($\alpha = 0$ if $r = 2$) and let

$$d\mu_{r,\alpha}(x) = Z_{r,\alpha}^{-1} \exp(-|x|^r (\log(\gamma + |x|))^\alpha) dx$$

$$\mu_{r,\alpha,n} = \mu_{p,\alpha}^{\otimes n},$$

where $Z_{r,\alpha}^{-1}$ is chosen to ensure that $\mu_{r,\alpha}(\mathbb{R}) = 1$. The corresponding results are apparently new and give fractional Sobolev inequalities, that just like the usual logarithmic Sobolev inequalities of [69], exhibit logarithmic gains of integrability that are directly related to the corresponding isoperimetric profiles. For example, if $\alpha = 0$, then the corresponding fractional Sobolev inequalities take the form. Let $1 \leq q < \infty$, $\theta \in (0, 1)$, then there exists an absolute constant $c > 0$, independent of the dimension, such that (cf. Theorem 24 below)

$$\left(\int_0^{1/2} |f|_{\mu_{r,0,n}}^* (t)^q \left(\log \frac{1}{t} \right)^{q\theta(1-1/r)} dt \right)^{1/q} \leq c \|f\|_{B_{L^q}^{\theta,q}(\mu_{r,0,n})}.$$

Likewise, for $q = \infty$ (cf. (3.4) below)

$$\sup_{t \in (0, \frac{1}{2})} \left(|f|_{\mu_{r,0,n}}^{**}(t) - |f|_{\mu_{r,0,n}}^*(t) \right) \left(\log \frac{1}{t} \right)^{(1-\frac{1}{r})\theta} \leq c \|f\|_{\dot{B}_{L^\infty}^{\theta, \infty}(\mu_{r,0,n})}.$$

We also explore the scaling of fractional inequalities for Gaussian Besov spaces based on exponential Orlicz spaces. We show that in this context the gain of integrability can be measured directly in the power of the exponential.

We will also consider the corresponding Morrey-Sobolev theorems for these measures.

2. Morrey-Sobolev theorem for Gaussian like measures

Let $\alpha \geq 0$, $r \in [1, 2]$ and $\gamma = \exp(2\alpha/(2-r))$ ($\alpha = 0$ if $r = 2$), and let $\mu_{r,\alpha}$ be the probability measure on \mathbb{R} defined by

$$d\mu_{r,\alpha}(x) = Z_{r,\alpha}^{-1} \exp(-|x|^r (\log(\gamma + |x|)^\alpha) dx = \varphi_{r,\alpha}(x) dx, x \in \mathbb{R},$$

where $Z_{r,\alpha}^{-1}$ is chosen to ensure that $\mu_{r,\alpha}(\mathbb{R}) = 1$. Then we let

$$\varphi_{\alpha,r}^n(x) = \varphi_{r,\alpha}(x_1) \cdots \varphi_{r,\alpha}(x_n), x \in \mathbb{R}^n,$$

and $\mu_{r,\alpha,n} = \mu_{r,\alpha}^{\otimes n}$. In other words

$$d\mu_{r,\alpha,n}(x) = \varphi_{r,\alpha}^n(x) dx.$$

In particular, $\mu_{2,0,n} = \gamma_n$ (Gaussian measure)

It is known that the isoperimetric problem for $\mu_{r,\alpha}$ is solved by half-lines (cf. [19] and [16]) and the isoperimetric profile is given by

$$I_{\mu_{r,\alpha}}(t) = \varphi(H^{-1}(\min(t, 1-t))) = \varphi(H^{-1}(t)), \quad t \in [0, 1],$$

where $H : \mathbb{R} \rightarrow (0, 1)$ is the increasing function given by

$$H(r) = \int_{-\infty}^r \varphi(x) dx.$$

Moreover (cf. [7] and [8]), there exist constants c_1, c_2 such that, for all $t \in [0, 1]$,

$$(2.1) \quad c_1 L_{\mu_{r,\alpha}}(t) \leq I_{\mu_{r,\alpha}}(t) \leq c_2 L_{\mu_{r,\alpha}}(t),$$

where

$$L_{\mu_{r,\alpha}}(t) = \min(t, 1-t) \left(\log \frac{1}{\min(t, 1-t)} \right)^{1-\frac{1}{r}} \left(\log \log \left(e + \frac{1}{\min(t, 1-t)} \right) \right)^{\frac{\alpha}{r}}.$$

Moreover, we have

$$(2.2) \quad I_{\mu_{r,\alpha,n}}(t) \simeq t \left(\log \frac{1}{t} \right)^{1-\frac{1}{r}} (\log \log(e + \frac{1}{t}))^{\alpha/r}, \text{ for } t \in (0, \frac{1}{2}).$$

For the rest of the section we shall let μ denote any of the measures $\mu_{r,\alpha,n}$ on \mathbb{R}^n . For a given r.i. space $X := X(\mathbb{R}^n, \mu)$, let $W_X^1(\mathbb{R}^n, \mu)$ be the classical Sobolev space endowed with the norm $\|u\|_{W_X^1(\mathbb{R}^n, \mu)} = \|u\|_X + \|\nabla u\|_X$. The homogeneous Sobolev space $\dot{W}_X^1(\mathbb{R}^n, \mu)$ is defined by means of the quasi norm $\|u\|_{\dot{W}_X^1} := \|\nabla u\|_X$.

The discussion of Chapter 5 - Section 2 applies and therefore we see that $W_{L_1}^1(\mathbb{R}^n, \mu)$ is invariant under truncation. Moreover, if $u \in W_{L_1}^1(\mathbb{R}^n, \mu)$ then the following co-area formula holds:

$$\int_{\mathbb{R}^n} |\nabla u(x)| d\mu(x) = \int_{\mathbb{R}^n} |\nabla u(x)| \varphi_{\alpha,p}^n(x) dx = \int_{-\infty}^{\infty} P_\mu(u > s) ds.$$

From here we see that inequalities [Chapter 5, (2.1), (2.4) and (2.5)] hold for all $W_{L_1}^1(\mathbb{R}^n, \mu)$ functions (of course, the rearrangements are now with respect to the measure μ). Finally, if we consider

$$K(t, f; X, \dot{W}_X^1) = \inf\{\|f - g\|_X + t \|g\|_{\dot{W}_X^1(\mathbb{R}^n, \mu)}\},$$

all the results that we have obtained in Chapter 3, remain true.

THEOREM 22. *If $\mu = \mu_{r,\alpha,n}$ then*

$$\dot{W}_X^1(\mathbb{R}^n, \mu) \not\subset L^\infty.$$

PROOF. By [69, Theorem 6]) the embedding $\dot{W}_X^1(\mathbb{R}^n, \mu) \subset L^\infty$ is equivalent to the existence of a positive constant $c > 0$, such that for all $f \in \bar{X}$, supported on $(0, \frac{1}{2})$ we have

$$\sup_{t \geq 0} \int_t^{1/2} \frac{|f(s)|}{L_{\mu_{r,\alpha,n}}(s)} \leq c \|f\|_{\bar{X}}.$$

In particular this implies that

$$\int_0^{1/2} \frac{ds}{L_{\mu_{r,\alpha,n}}(s)} \leq c.$$

But this is not possible since $1/L_{\mu_{r,\alpha,n}}(s) \notin L^1$. \square

It follows that the results of Chapter 5 cannot be applied directly to obtain the continuity of functions in the space \dot{W}_X^1 . Indeed, since $1/L_{\mu_{r,\alpha,n}}(s) \notin L^1$ implies that for any r.i. space X , $1/L_{\mu_{r,\alpha,n}}(s) \notin \bar{X}$. however since continuity is a local property we can show the following version of the Morrey-Sobolev theorem.

THEOREM 23. *Let $\mu = \mu_{r,\alpha,n}$, and let $X = X(\mathbb{R}^n, \mu)$ be a r.i space on (\mathbb{R}^n, μ) such that*

$$\left\| \frac{1}{\min(1, 1-t)^{1-1/n}} \right\|_{\bar{X}'} < \infty.$$

Then every function in $\dot{W}_X^1(\mathbb{R}^n, \mu)$ is essentially continuous.

PROOF. Let $f \in \dot{W}_X^1(\mathbb{R}^n, \mu)$ and let $B \subset \mathbb{R}^n$ be an arbitrary ball with Lebesgue measure equal to 1. We prove that f is continuous on B . Note that $f\chi_B \in \dot{W}_X^1(B, \mu)$, i.e.

$$\|\nabla f\chi_B\|_X < \infty.$$

Let m be the Lebesgue measure on \mathbb{R}^n , it is plain that

$$c_B m\{x \in B : |\nabla f| > t\} \leq \mu\{x \in B : |\nabla f| > t\} \leq C_B m\{x \in B : |\nabla f| > t\}$$

where $c_B = \inf_{x \in B} \varphi_{\alpha,p}^n(x)$ and $C_B = \max_{x \in B} \varphi_{\alpha,p}^n(x)$. Therefore,

$$c_Q \|\nabla f\chi_B\|_{X(B,m)} \leq \|\nabla f\chi_B\|_{X(\mathbb{R}^n, \mu)} \leq C_Q \|\nabla f\chi_B\|_{X(B,m)}.$$

Consequently, $f\chi_B \in \dot{W}_X^1(B, m)$, and by Theorem 14, $f\chi_B \in C(B)$. \square

3. Embeddings of Gaussian Besov spaces

Let μ denote any of the measures $\mu_{r,\alpha,n}$ on \mathbb{R}^n . We consider the Besov spaces $\dot{B}_X^{\theta,q}(\mu), B_X^{\theta,q}(\mu)$ can be defined using real interpolation (cf. [14], [93]). In other words for $1 \leq q \leq \infty, \theta \in (0, 1)$, and let

$$\begin{aligned} \dot{B}_X^{\theta,q}(\mu) &= \{f : \|f\|_{B_X^{\theta,q}(\mu)} = \|f\|_{\dot{B}_X^{\theta,q}(\mu)} < \infty\}, \\ (3.1) \quad B_X^{\theta,q}(\mu) &= \{f : \|f\|_{B_X^{\theta,q}(\mu)} = \|f\|_{\dot{B}_X^{\theta,q}(\mu)} + \|f\|_X < \infty\} \end{aligned}$$

where

$$\|f\|_{\dot{B}_X^{\theta,q}(\mu)} = \begin{cases} \left(\int_0^1 \left(K \left(s, f; X(\mu), \dot{W}_X^1(\mu) \right) s^{-\theta} \right)^q \frac{ds}{s} \right)^{1/q} & \text{if } q < \infty \\ \sup_s \left(K \left(s, f; X(\mu), \dot{W}_X^1(\mu) \right) s^{-\theta} \right) & \text{if } q = \infty. \end{cases}$$

The embeddings we prove in this section will follow from

$$(3.2) \quad f^{**}(t) - f^*(t) \leq c \frac{K \left(\frac{t}{I_\mu(t)}, f; X(\mu), \dot{W}_X^1(\mu) \right)}{\phi_X(t)}, f \in X + \dot{W}_X^1.$$

To simplify the presentation we shall state and prove our results only for the Gaussian measures $\mu_{r,0,n}, r \in (1, 2]$, which include the most important examples: Gaussian measures and the so called interpolation measures between exponential and Gaussian.

THEOREM 24. *Let $1 \leq q < \infty, \theta \in (0, 1), r \in (1, 2]$. Then there exists an absolute constant $c = c(q, \theta, r) > 0$ such that,*

$$(3.3) \quad \left\{ \int_0^{1/2} |f|_{\mu_{r,0,n}}^*(t)^q \left(\log \frac{1}{t} \right)^{q\theta(1-1/r)} dt \right\}^{1/q} \leq c \|f\|_{B_{L^q}^{\theta,q}(\mu_{r,0,n})}.$$

Let $q = \infty$, then there exists an absolute constant $c = c(\theta, r) > 0$ such that

$$(3.4) \quad \sup_{t \in (0, \frac{1}{2})} \left(|f|_{\mu_{r,0,n}}^{**}(t) - |f|_{\mu_{r,0,n}}^*(t) \right) \left(\log \frac{1}{t} \right)^{(1-\frac{1}{r})\theta} \leq c \|f\|_{\dot{B}_{L^\infty}^{\theta,\infty}(\mu_{r,0,n})}.$$

PROOF. We shall let $\mu =: \mu_{r,0,n}, K(s, f) := K(s, f; L^q(\mu), \dot{W}_{L^q}^1(\mu))$. Suppose that $1 \leq q < \infty$. We start by rewriting the term we want to estimate

$$\begin{aligned} & \left\{ \int_0^{1/2} |f|_\mu^*(t)^q \left(\log \frac{1}{t} \right)^{q\theta(1-1/r)} dt \right\}^{1/q} \\ & \leq \left\{ \int_0^{1/2} |f|_\mu^*(t)^q \left(\int_t^{1/2} \left(\log \frac{1}{s} \right)^{q\theta(1-1/r)-1} \frac{ds}{s} + (\log 2)^{q\theta(1-1/r)} \right) dt \right\}^{1/q} \\ & \leq \left\{ \int_0^{1/2} \left(\log \frac{1}{s} \right)^{q\theta(1-1/r)-1} \frac{1}{s} \int_0^s |f|_\mu^*(t)^q dt ds \right\}^{1/q} + \left\{ \int_0^{1/2} |f|_\mu^*(t)^q dt \right\}^{1/q} \\ & = (I) + (II) \end{aligned}$$

The term (II) is under control since

$$(II) \leq \|f\|_{L^q} \leq \|f\|_{B_{L^q}^{\theta,q}(\mu)}.$$

To estimate (I) we first note that the elementary inequality¹: $|x|^q \leq 2^{q-1}(|x-y|^q + |y|^q)$, yields

$$\frac{1}{s} \int_0^s |f|_\mu^*(t)^q dt \preceq \frac{1}{s} \int_0^s (f_\mu^*(t) - f_\mu^*(s))^q dt + f_\mu^*(s)^q.$$

Consequently,

$$\begin{aligned} (I) &\preceq \left\{ \int_0^{1/2} \left(\log \frac{1}{s} \right)^{q\theta(1-1/r)-1} \left(\frac{1}{s} \int_0^s (|f|_\mu^*(t) - |f|_\mu^*(s))^q dt \right) ds \right\}^{1/q} \\ &\quad + \left\{ \int_0^{1/2} \left(\log \frac{1}{s} \right)^{q\theta(1-1/r)-1} |f|_\mu^*(s)^q ds \right\}^{1/q} \\ &= (I_1) + (I_2), \text{ say.} \end{aligned}$$

To control (I_1) we first use Example 2.15 in Chapter 3 to estimate the inner integral as follows

$$\frac{1}{s} \int_0^s (|f|_\mu^*(t) - |f|_\mu^*(s))^q dt \preceq \frac{1}{s} \left(K((\log \frac{1}{s})^{1/r-1}, f) \right)^q.$$

Thus,

$$(I_1) \preceq \left\{ \int_0^{1/2} \left(\left(\log \frac{1}{s} \right)^{\theta(1-1/r)} \left(K((\log \frac{1}{s})^{1/r-1}, f) \right)^q \frac{ds}{s \log \frac{1}{s}} \right) \right\}^{1/q}$$

The change of variables $u = (\log \frac{1}{s})^{1/r-1}$ then yields

$$(I_1) \preceq \|f\|_{B_{L^q}^{\theta, q}(\mu)}.$$

It remains to estimate (I_2) . We write

$$(I_2) \leq \left\{ \int_0^{1/2} \left(\log \frac{1}{s} \right)^{q\theta(1-1/r)-1} |f|_\mu^{**}(s)^q ds \right\}^{1/q},$$

then, using the fundamental theorem of calculus, we have

$$\begin{aligned} (I_2) &\leq \left\{ \int_0^{1/2} \left(\log \frac{1}{s} \right)^{q\theta(1-1/r)-1} \left(\int_t^{1/2} (|f|_\mu^{**}(s) - |f|_\mu^*(s)) \frac{ds}{s} + |f|_\mu^{**}(1/2) \right)^q ds \right\}^{1/q} \\ &\leq \left\{ \int_0^{1/2} \left(\left(\log \frac{1}{s} \right)^{\theta(1-1/r)-1/q} \int_t^{1/2} (|f|_\mu^{**}(s) - |f|_\mu^*(s)) \frac{ds}{s} \right)^q ds \right\}^{1/q} \\ &\quad + |f|_\mu^{**}(1/2) \left\{ \int_0^{1/2} \left(\log \frac{1}{t} \right)^{q\theta(1-1/r)} dt \right\}^{1/q} \\ &= (A) + (B), \text{ say.} \end{aligned}$$

To use the Hardy logarithmic inequality of [12, (6.7)] we first write

$$(A) = \left\{ \int_0^{1/2} \left(\left(\log \frac{1}{s} \right)^{\theta(1-1/r)-1/q} s^{1/q} \int_t^{1/2} (|f|_\mu^{**}(s) - |f|_\mu^*(s)) \frac{ds}{s} \right)^q \frac{ds}{s} \right\}^{1/q}$$

¹Which follows readily by Jensen's inequality.

and then find that

$$(A) \preceq \left\{ \int_0^{1/2} \left((|f|_{\mu}^{**}(s) - |f|_{\mu}^*(s)) s^{1/q} \left(\log \frac{1}{s} \right)^{\theta(1-1/r)-1/q} \right)^q \frac{ds}{s} \right\}^{1/q}.$$

Now we use the fact that in the region of integration $s^{1/q} \leq 1$, combined with (3.2) and (2.2), to conclude that

$$\begin{aligned} & \left\{ \int_0^{1/2} \left((|f|_{\mu}^{**}(s) - |f|_{\mu}^*(s)) s^{1/q} \left(\log \frac{1}{s} \right)^{\theta(1-1/r)-1/q} \right)^q \frac{ds}{s} \right\}^{1/q} \\ & \preceq \left\{ \int_0^{1/2} \left(K \left(\left(\log \frac{1}{s} \right)^{\frac{1}{r}-1}, f \right) \right)^q \left(\log \frac{1}{s} \right)^{q\theta(1-1/r)} \frac{ds}{s \left(\log \frac{1}{s} \right)} \right\}^{1/q} \\ & \simeq \left\{ \int_0^{1/2} (K(u, f))^q u^{-\theta q} \frac{du}{u} \right\}^{1/q} \quad (\text{change of variables } u = \left(\log \frac{1}{s} \right)^{\frac{1}{r}-1}) \\ & \leq \|f\|_{B_{L^q}^{\theta, q}(\mu)}. \end{aligned}$$

Finally it remains to estimate (B) :

$$\begin{aligned} (B) &= 2 \left(\frac{1}{2} |f|_{\mu}^{**}(1/2) \right) \left\{ \int_0^{1/2} t^{1/2} \left(\log \frac{1}{t} \right)^{q\theta(1-1/r)-1} \frac{dt}{t^{1/2}} \right\}^{1/q} \\ &\leq 2 \|f\|_{L^1} \left(\sup_{t \in (0, 1/2)} t^{1/2} \left(\log \frac{1}{t} \right)^{q\theta(1-1/r)-1} \right) \left\{ \int_0^{1/2} \frac{dt}{t^{1/2}} \right\} \\ &\leq \|f\|_{L^1} \leq \|f\|_{L^q} \\ &\leq \|f\|_{B_{L^q}^{\theta, q}(\mu)}. \end{aligned}$$

We consider now the case $q = \infty$. We apply (3.2), observing that for $X = L^\infty$, $\phi_{L^\infty}(t) = 1$, and obtain that for $t \in (0, \frac{1}{2})$,

$$\begin{aligned} |f|_{\mu_r}^{**}(t) - |f|_{\mu_r}^*(t) &\preceq K \left(\left(\log \frac{1}{t} \right)^{\frac{1}{r}-1}, f \right) \\ &= K \left(\left(\log \frac{1}{t} \right)^{\frac{1}{r}-1}, f \right) \left(\log \frac{1}{t} \right)^{-(\frac{1}{r}-1)\theta} \left(\log \frac{1}{t} \right)^{(\frac{1}{r}-1)\theta} \\ &\leq \left(\log \frac{1}{t} \right)^{(\frac{1}{r}-1)\theta} \left(\sup_u (K(u, f) u^{-\theta}) \right) \\ &= \left(\log \frac{1}{t} \right)^{(\frac{1}{r}-1)\theta} \|f\|_{\dot{B}_{L^\infty}^{\theta, \infty}(\mu_r)}, \end{aligned}$$

as we wished to show. \square

REMARK 9. *Gaussian measure corresponds to $r = 2$, in this case, for $q = 2$, (3.3) yields the logarithmic Sobolev inequality*

$$\left\{ \int_0^{1/2} |f|_{\gamma_n}^*(t)^2 \left(\log \frac{1}{t} \right)^\theta dt \right\}^{1/2} \leq c \|f\|_{B_{L^2}^{\theta, 2}(\gamma_n)}, \theta \in (0, 1).$$

Formally the case $\theta = 1$ corresponds to an L^2 Logarithmic Sobolev inequality, while the case $\theta = 0$, corresponds to the trivial $L^2 \subset L^2$ embedding. One could formally approach such inequalities by complex interpolation (cf. [6] as well as the calculations provided in [75])

$$[L^2, \dot{W}_{L^2}^1]_\theta \subset [L^2, L^2 \text{Log} L]_\theta = L^2(\text{Log} L)^\theta.$$

The case $r = 2$, $q = 1$, corresponds to a fractional version of Ledoux's inequality (cf. [59]). Besides providing a unifying approach our method can be applied to deal with more general domains and measures.

REMARK 10. When $q = \infty$ the inequality (3.4) reflects a refined estimate of the exponential integrability of f . In particular, note that the case $\theta = 1$, formally gives the inequality (cf. [17] and the references therein)

$$\|f\|_{WL^\infty(\gamma_n)} = \sup_{t \in (0, \frac{1}{2})} \left(|f|_{\gamma_n}^{**}(t) - |f|_{\gamma_n}^*(t) \right) \leq c \|f\|_{\dot{W}_{eL^2}^1(\gamma_n)}.$$

The previous inequality can be proved readily using

$$|f|_{\gamma_n}^{**}(t) - |f|_{\gamma_n}^*(t) \leq c \frac{1}{(\log \frac{1}{t})^{1/2}} |\nabla f|_{\gamma_n}^{**}(t), \quad t \in (0, 1/2).$$

REMARK 11. Using the transference principle of [69] the Gaussian results can be applied to derive results related to the dimensionless Sobolev inequalities on Euclidean cubes studied by Krbeč-Schmeisser (cf. [56], [57]) and Triebel [94].

4. Exponential Classes

There is a natural connection between Gaussian measure and the exponential class e^{L^2} . Likewise, this is also true with more general exponential measures and other exponential spaces. Although there are many nice inequalities associated with this topic that follow from our theory, we will not develop the matter in great detail here. Instead, we shall only give a flavor of possible results by considering Besov embeddings connected with the Sobolev space $\dot{W}_{eL^2}^1 := \dot{W}_{eL^2}^1(\mathbb{R}^n, \gamma_n)$.

In this setting (3.2) takes the form

$$|f|_{\gamma_n}^{**}(t) - |f|_{\gamma_n}^*(t) \leq c \frac{K((\log \frac{1}{t})^{-\frac{1}{2}}, f; e^{L^2}, \dot{W}_{eL^2}^1)}{\phi_{eL^2}(t)}, \quad t \in (0, \frac{1}{2}).$$

Now, since $\phi_{eL^2}(t) = (\log \frac{1}{t})^{-\frac{1}{2}}$, $t \in (0, \frac{1}{2})$ we have

$$\begin{aligned} \left(|f|_{\gamma_n}^{**}(t) - |f|_{\gamma_n}^*(t) \right) &\leq c K\left(\left(\log \frac{1}{t}\right)^{-\frac{1}{2}}, f; e^{L^2}, \dot{W}_{eL^2}^1\right) \left(\log \frac{1}{t}\right)^{\frac{1}{2}} \\ &\leq c \|f\|_{\dot{B}_{eL^2, \infty}^1}, \end{aligned}$$

that is

$$\|f\|_{WL^\infty(\gamma_n)} \leq c \|f\|_{\dot{B}_{eL^2, \infty}^1}.$$

More generally,

$$\begin{aligned} \left(|f|_{\gamma_n}^{**}(t) - |f|_{\gamma_n}^*(t) \right) \left(\log \frac{1}{t}\right)^{-\frac{1}{2} + \frac{\theta}{2}} &\leq c K\left(\left(\log \frac{1}{t}\right)^{-\frac{1}{2}}, f; e^{L^2}, \dot{W}_{eL^2}^1\right) \left(\log \frac{1}{t}\right)^{\frac{\theta}{2}} \\ &\leq c \|f\|_{\dot{B}_{eL^2, \infty}^\theta}, \end{aligned}$$

which shows directly the improvement on the exponential integrability in the $\dot{B}_{eL^2, \infty}^\theta$ scale.

CHAPTER 7

On limiting Sobolev embeddings and BMO

1. Introduction and Summary

The discussion in this chapter is connected with the role of BMO in some limiting Sobolev inequalities. We start by reviewing some definitions and then proceed to describe some Sobolev inequalities which follow readily from our symmetrization inequalities and will be relevant for our discussion.

Let (Ω, d, μ) be a probability metric measure space, and let $f : \Omega \rightarrow \mathbb{R}$ be an integrable function. The space $BMO(\Omega) = BMO$, introduced by John-Nirenberg, is defined by the condition

$$\|f\|_{BMO_*} = \sup_Q \left\{ \inf_c \left(\frac{1}{\mu(Q)} \int_Q |f - c| d\mu \right) : Q \text{ open ball contained in } \Omega \right\} < \infty.$$

In fact, it is enough to consider averages

$$\|f\|_* \simeq \sup_Q \left\{ \frac{1}{\mu(Q)} \int_Q |f - f_Q(f)| d\mu : Q \text{ open ball contained in } \Omega \right\} < \infty,$$

where $f_Q(f) = \frac{1}{\mu(Q)} \int_Q f d\mu$. To obtain a norm we may set

$$\|f\|_{BMO} = \|f\|_* + \|f\|_{L^1}.$$

REMARK 12. *Under suitable assumptions on the metric space, for example for homogeneous metric spaces, one can also control $\|f\|_*$ through the use of maximal operators (cf. [38], [26], [1]). Let*

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f - f_Q(f)| d\mu,$$

where the sup is taken over all open balls containing x , then we have

$$\|f\|_* \simeq \|f^\#\|_\infty.$$

Let $\theta \in (0, 1)$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. Consider the Besov spaces $\dot{b}_{p,q}^\theta(\Omega)$ (resp. $b_{p,q}^\theta(\Omega)$), defined by

$$(1.1) \quad \|f\|_{\dot{b}_{p,q}^\theta(\Omega)} = \left(\int_0^{1/2} (t^{-\theta} K(t, f, L^p(\Omega), w_{L^p}(\Omega))^q \frac{dt}{t} \right)^{1/q}$$

$$\|f\|_{b_{p,q}^\theta(\Omega)} = \|f\|_{\dot{b}_{p,q}^\theta(\Omega)} + \|f\|_{L^p}.$$

For ready comparison with classical embedding theorems, from now on in this section, we shall consider probability metric spaces (Ω, d, μ) such that the corresponding isoperimetric profiles satisfy

$$t^{1-1/n} \preceq I_\Omega(t), \quad t \in (0, 1/2).$$

We now recall the definition of the $L^{p,q}$ spaces. Moreover, in order to incorporate in a meaningful way the limiting cases that correspond to the index $p = \infty$, we also recall the definition of the modified $L^{[p,q]}$ spaces¹. Let $1 \leq p < \infty, 1 \leq q \leq \infty$ (cf. [11], [9]), and let²

$$L^{p,q}(\Omega) = \left\{ f : \|f\|_{L^{p,q}} = \left(\int_0^{\mu(\Omega)} \left(|f|_\mu^*(t) s^{1/p} \right)^q \frac{ds}{s} \right)^{1/q} < \infty \right\}.$$

For $1 \leq p \leq \infty, 1 \leq q \leq \infty$, we let

$$L^{[p,q]}(\Omega) = \left\{ f : \|f\|_{L^{[p,q]}} = \left(\int_0^{\mu(\Omega)} \left((|f|_\mu^{**}(t) - |f|_\mu^*(t)) s^{1/p} \right)^q \frac{ds}{s} \right)^{1/q} < \infty \right\}.$$

It is known that (cf. [70] and the references therein)

$$L^{p,q}(\Omega) = L^{[p,q]}(\Omega), \text{ for } 1 \leq p < \infty, 1 \leq q \leq \infty.$$

Then, under our current assumptions on the isoperimetric profile of Ω , Theorem 7 states that

$$(1.2) \quad |f|_\mu^{**}(t) - |f|_\mu^*(t) \leq c \frac{K(t^{1/n}, f; L^p(\Omega), S_{L^p}(\Omega))}{t^{1/p}}, \quad t \in (0, 1/2).$$

The following basic version of the Sobolev embedding follows readily

PROPOSITION 3.

$$(1.3) \quad b_{p,q}^\theta(\Omega) \subset L^{\bar{p},q}(\Omega), \text{ where } \frac{1}{\bar{p}} = \frac{1}{p} - \frac{\theta}{n}, \quad \theta \in (0, 1), \quad 1 \leq q \leq \infty, \quad \theta p \leq n.$$

PROOF. Indeed, from the relationship between the indices and (1.2) we can write

$$\left(|f|_\mu^{**}(t) - |f|_\mu^*(t) \right) t^{1/\bar{p}} \preceq t^{-\frac{\theta}{n}} K(t^{1/n}, f; L^p(\Omega), S_{L^p}(\Omega)), \quad t \in (0, 1/2).$$

If $q = \infty$, (1.3) follows taking supremum on both sides of the inequality above. Likewise, if $q < \infty$, then the desired result follows raising both sides to the power q and integrating from 0 to 1/2. In reference to the role of the $L[\infty, q]$ spaces here let us remark that, in the limiting case $\theta p = n$, we have $\bar{p} = \infty$. \square

We consider the limiting case, $\theta = \frac{n}{p}$, $p > n$, in more detail. In this case (1.3) reads (cf. [64])

$$b_{p,q}^{n/p}(\Omega) \subset L^{[\infty,q]}(\Omega), \quad p > n, \quad 1 \leq q \leq \infty.$$

Note that when $q = 1$, $L^{[\infty,1]}(\Omega) = L^\infty(\Omega)$, and we recover the well known result (for Euclidean domains),

$$(1.4) \quad b_{p,1}^{n/p}(\Omega) \subset L^\infty(\Omega).$$

On the other hand, when $q = \infty$, from (1.3) we only get

$$(1.5) \quad \dot{b}_{p,\infty}^{n/p}(\Omega) \subset L^{[\infty,\infty]}(\Omega).$$

¹The $L(p, q)$ and $L[p, q]$ spaces are equivalent for $p < \infty$.

²with the usual modifications when $q = \infty$.

In the Euclidean world better results are known. Recall that given a domain $\Omega \subset \mathbb{R}^n$ the Besov spaces $\dot{B}_{p,q}^\theta(\Omega)$ (resp. $B_{p,q}^\theta(\Omega)$), are defined by

$$(1.6) \quad \|f\|_{\dot{B}_{p,q}^\theta(\Omega)} = \left(\int_0^{|\Omega|} \left(t^{-\theta} K(t, f, L^p(\Omega), \dot{W}_{L^p}^1(\Omega)) \right)^q \frac{dt}{t} \right)^{1/q}$$

$$\|f\|_{B_{p,q}^\theta(\Omega)} = \|f\|_{\dot{B}_{p,q}^\theta(\Omega)} + \|f\|_{L^p}.$$

Indeed, for smooth domains, we have a better result than (1.4), namely

$$(1.7) \quad B_{p,1}^{n/p}(\Omega) \subset C(\Omega),$$

and, moreover, it is well known that (cf. [20])

$$(1.8) \quad \dot{B}_{p,\infty}^{n/p}([0,1]^n) \subset BMO([0,1]^n).$$

We note that since we have³ $BMO([0,1]^n) \subset L^{[\infty,\infty]}$: i.e.

$$\sup_t (|f|^{**}(t) - |f|^*(t)) \leq C \|f\|_*,$$

then (1.8) is stronger than (1.5).

In Chapter 4 we have shown that for Sobolev and Besov spaces that are based on metric spaces with the relative isoperimetric property, the rearrangement inequality (1.4) self-improves to (1.7). In this chapter we discuss the corresponding method of self-improvement that leads from (1.5) to (1.8). In fact, our methods are connected with a characterization of BMO using rearrangements which is very close, in spirit, to the one provided by John [53] and Stromberg [89].

Finally, re-interpreting BMO as a limiting Lip space we were lead to an analog of [Chapter 3, (1.1)] which we now describe. In \mathbb{R}^n we argue that the natural replacement of [(1.2), Chapter 1] involving the space BMO is given by the Bennett-DeVore-Sharp inequality (cf. [11], [13], [1], [2])

$$(1.9) \quad |f|^{**}(t) - |f|^*(t) \leq c(f^\#)^*(t), \quad 0 < t < \frac{|B|}{6}, \text{ where } B \text{ is a ball on } \mathbb{R}^n.$$

Variants of this inequality are known to hold in more general contexts. For our purposes here the following inequality will suffice

$$(1.10) \quad |f|_\mu^{**}(t) - |f|_\mu^*(t) \leq C \|f\|_*, \quad 0 < t < \mu(\Omega).$$

We shall therefore assume in our discussion that (Ω, d, μ) is a metric space with finite measure such that there exists a constant $C > 0$ such that (1.10) holds for all $f \in BMO$. For example, in [84, see (3.8)] it is shown that (1.10) holds for doubling measures on Euclidean domains. More general results can be found in [1].

Assuming the validity of (1.10), and using the method of the proof of Theorem 7, we will show below (see Theorem 27) that if $X(\Omega)$ is a r.i. space then we have⁴

$$(1.11) \quad |f|^{**}(t) - |f|^*(t) \leq c \frac{K(\phi_X(t), f; X(\Omega), BMO(\Omega))}{\phi_X(t)}, \quad 0 < t < \mu(\Omega).$$

This result should be compared with Theorem 7 above. For perspective, we now show a different road to a special case of (1.11). Recall that for Euclidean domains

³This is an easy consequence of (1.9) below.

⁴On \mathbb{R}^n (1.11) is known and can be obtained by combining (1.9) with [13, theorem 8.8].

it is shown in [13, (8.11)] that

$$\frac{K(t, f; L^1, BMO)}{t} \simeq (f^\#)^*(t).$$

Combining this inequality with (1.9), we obtain a different approach to (1.11) in the special case $X = L^1$, at least when t is close to zero.

1.1. Self Improving inequalities and BMO . The self improvement of inequalities discussed in the introduction of this chapter follow directly from the next theorem.

THEOREM 25. (i) Let $p > n$. Then there exists $c(n, p) > 0$ such that

$$\sup_{Q \subset [0,1]^n, Q \text{ open cube}} \{(|f\chi_Q|)^{**}(|Q|) - (|f\chi_Q|)^*(|Q|)\} \leq c(n, p) \|f\|_{\dot{B}_{p,\infty}^{n/p}([0,1]^n)}.$$

(ii) There exists a constant $C > 0$ such that for all $f \geq 0$, it holds

$$\|f\|_{BMO([0,1]^n)} \leq C \sup_{Q \subset [0,1]^n, Q \text{ open cube}} \{(f\chi_Q)^{**}(|Q|) - (f\chi_Q)^*(|Q|)\}.$$

PROOF. (i) Let $Q \subset [0,1]^n$ and open cube, then (cf. (2.6), Chapter 5),

$$I_Q(t) \geq c(n) \min(s, (|Q| - s))^{\frac{n-1}{n}}, \quad 0 < s < |Q|.$$

Given $f \in \dot{B}_{p,\infty}^{n/p}([0,1]^n)$, then $f\chi_Q \in \dot{B}_{p,\infty}^{n/p}(Q)$ with

$$\|f\chi_Q\|_{\dot{B}_{p,\infty}^{n/p}(Q)} \leq \|f\|_{\dot{B}_{p,\infty}^{n/p}([0,1]^n)}.$$

Applying Theorem 10 we obtain, for $0 < t < |Q|$,

$$(|f\chi_Q|)^{**}(t) - (|f\chi_Q|)^*(t) \leq \frac{K\left(\psi_Q(t), f\chi_Q; L^p(Q), \dot{W}_{L^p}^1(Q)\right)}{t^{1/p}},$$

where

$$\psi_Q(t) = t^{1/p-1} \left\| \frac{s}{I_Q(s)} \chi_{(0,t)}(s) \right\|_{L^{p'}(Q)}.$$

Since $p > n$, an elementary computation shows that

$$\left\| \frac{1}{\min(s, (|Q| - s))^{\frac{n-1}{n}}} \chi_{(0,t)}(s) \right\|_{L^{p'}(Q)} \leq c_{(n,p)} t^{\frac{1}{n} - \frac{1}{p}}, \quad 0 < t < |Q|.$$

Thus,

$$\begin{aligned} \psi_Q(t) &\leq \frac{t^{1/p-1}}{c(n)} \left\| \frac{s}{\min(s, (|Q| - s))^{\frac{n-1}{n}}} \chi_{(0,t)}(s) \right\|_{L^{p'}(Q)} \\ &\leq \frac{t^{1/p}}{c(n)} \left\| \frac{s}{\min(s, (|Q| - s))^{\frac{n-1}{n}}} \chi_{(0,t)}(s) \right\|_{L^{p'}(Q)} \\ &\leq c_{(n,p)} t^{\frac{1}{n}}, \quad 0 < t < |Q|. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{K\left(\psi_Q(t), |f| \chi_Q; L^p(Q), \dot{W}_{L^p}^1(Q)\right)}{t^{1/p}} &\leq \frac{K\left(c_{(n,p)} t^{1/n}, |f| \chi_Q; L^p(Q), \dot{W}_{L^p}^1(Q)\right)}{t^{1/p}} \\ &\leq \max(1, c_{(n,p)}) \frac{K\left(t^{1/n}, |f| \chi_Q; L^p(Q), \dot{W}_{L^p}^1(Q)\right)}{t^{1/p}}. \end{aligned}$$

Combining these results:

$$\begin{aligned} &(|f \chi_Q|)^{**}(|Q|) - (|f \chi_Q|)^*(|Q|) \\ &\leq \max(1, c_{(n,p)}) \sup_{0 < t < |Q|} \frac{K\left(t^{1/n}, f \chi_Q; L^p(Q), \dot{W}_{L^p}^1(Q)\right)}{t^{1/p}} \\ &= \max(1, c_{(n,p)}) \sup_{0 < t < |Q|^n} \frac{K\left(t, f \chi_Q; L^p(Q), \dot{W}_{L^p}^1(Q)\right)}{t^{n/p}} \\ &= \max(1, c_{(n,p)}) \|f \chi_Q\|_{\dot{B}_{p\infty}^{n/p}(Q)} \\ &\leq \max(1, c_{(n,p)}) \|f\|_{\dot{B}_{p\infty}^{n/p}([0,1]^n)}. \end{aligned}$$

(ii) For $t \in (0, 1)$ let us write⁵

$$\begin{aligned} t(|f|^{**}(t) - |f|^*(t)) &= \int_0^t (|f|^*(x) - |f|^*(t)) dx \\ &= \int_0^\infty [|f|^*(x) - |f|^*(t)]_+ dx \\ &= \int_{\{s: f(s) > f^*(t)\}} [f(s) - |f|^*(t)]_+ d^n s. \end{aligned}$$

Fix a cube Q and apply the preceding equality to $f \chi_Q$ and $t = |Q|$:

$$\begin{aligned} &|Q| ((|f \chi_Q|)^{**}(|Q|) - (|f \chi_Q|)^*(|Q|)) \\ &= \int_{\{s \in Q: f \chi_Q(s) > (f \chi_Q)^*(|Q|)\}} [f \chi_Q(s) - (|f \chi_Q|)^*(|Q|)] d^n s. \end{aligned}$$

To estimate the right hand side from below we observe that

$$\frac{1}{|Q|} \int_Q f(x) d^n x = \frac{1}{|Q|} \int_0^{|Q|} (|f \chi_Q|)^*(s) ds \geq (|f \chi_Q|)^*(|Q|),$$

therefore

$$f(s) - (|f \chi_Q|)^*(|Q|) \geq f(s) - \frac{1}{|Q|} \int_Q f(x) d^n x.$$

⁵We shall use the convention of denoting the Lebesgue measure on $[0, 1]$ by dx , while $d^n x$ denotes the corresponding Lebesgue measure on $[0, 1]^n$.

Consequently,

$$\begin{aligned}
& \int_{\{s \in Q: f(s) > (f\chi_Q)^*(|Q|)\}} [f(s) - (f\chi_Q)^*(|Q|)] d^n s \\
& \geq \int_{\{s \in Q: f(s) > \frac{1}{|Q|} \int_Q f\}} [f(s) - (f\chi_Q)^*(|Q|)] d^n s \\
& \geq \int_{\{x \in Q: f(s) > \frac{1}{|Q|} \int_Q f\}} \left[f(s) - \frac{1}{|Q|} \int_Q f \right] d^n s.
\end{aligned}$$

We will verify in a moment that

$$\begin{aligned}
(1.12) \quad & \frac{1}{|Q|} \int_{\{x \in Q: f(s) > \frac{1}{|Q|} \int_Q f\}} \left[f(s) - \frac{1}{|Q|} \int_Q f \right] d^n s \\
& = \frac{1}{2} \frac{1}{|Q|} \int_Q \left| f(s) - \frac{1}{|Q|} \int_Q f \right| d^n s.
\end{aligned}$$

Combining (1.12) with the previous estimates we see that

$$((|f\chi_Q|)^{**}(|Q|) - (|f\chi_Q|)^*(|Q|)) \geq \frac{1}{2} \frac{1}{|Q|} \int_Q \left| f(s) - \frac{1}{|Q|} \int_Q f \right| d^n s.$$

Hence

$$\begin{aligned}
\|f\|_{BMO} &= \sup_Q \frac{1}{|Q|} \int_Q \left| f(s) - \frac{1}{|Q|} \int_Q f \right| d^n s \\
&\leq 2 \sup_Q ((|f\chi_Q|)^{**}(|Q|) - (|f\chi_Q|)^*(|Q|)),
\end{aligned}$$

as we wished to show.

It remains to see (1.12). Let us write in what follows $\frac{1}{|Q|} \int_Q f = f_Q$, then

$$\int_{\{x \in Q: f(s) > f_Q\}} [f(s) - f_Q] d^n s + \int_{\{x \in Q: f(s) < f_Q\}} [f(s) - f_Q] d^n s = 0.$$

Therefore

$$\int_{\{x \in Q: f(s) > f_Q\}} [f(s) - f_Q] d^n s = \int_{\{x \in Q: f(s) < f_Q\}} [f_Q - f(s)] d^n s$$

and

$$\begin{aligned}
\int_Q |f(s) - f_Q| d^n s &= \int_{\{x \in Q: f(s) > f_Q\}} [f(s) - f_Q] d^n s + \int_{\{x \in Q: f(s) < f_Q\}} [f_Q - f(s)] d^n s \\
&= 2 \int_{\{x \in Q: f(s) > f_Q\}} [f(s) - f_Q] d^n s.
\end{aligned}$$

□

The epilogue to our current discussion is now given by

COROLLARY 1. *Suppose that $f \in \dot{B}_{p,\infty}^{n/p}([0,1]^n)$ then $f \in BMO([0,1]^n)$.*

PROOF. Let $\phi_+(f) = f^+ = \max\{f, 0\}$ and $\phi_-(f) = -\min\{f, 0\}$. Then, for $T = \phi_+$, or $T = \phi_-$, we have $K(t, T(f), L^p, \dot{W}_{L^p}^1) \leq K(t, f; L^p, \dot{W}_{L^p}^1)$. Consequently, $\phi_+(f)$ and $\phi_-(f) \in \dot{B}_{p,\infty}^{n/p}([0,1]^n)$. Hence, by the previous theorem, $\phi_+(f)$ and $\phi_-(f) \in BMO$, and, $f = \phi_+(f) - \phi_-(f) \in BMO$. □

2. On the John-Stromberg characterization of BMO

Our discussion in this chapter is closely connected with a characterization of $BMO(Q)$ using rearrangements due to John [53] and Stromberg [89]. Let $\lambda \in (0, \frac{1}{2}]$, then

$$\|f\|_{BMO_*} \simeq \sup_{Q \subset Q_0} \inf_{c \in \mathbb{R}} ((f - c)\chi_Q)^*(\lambda|Q|).$$

See also Jawerth-Torchinsky [52], Lerner [61], [30], and the references therein.

DEFINITION 3. Let (Ω, d, μ) be a probability metric space, let $f : \Omega \rightarrow \mathbb{R}$ be an integrable function. We say that $m(f)$ is a median value if

$$\mu\{f > m(f)\} \leq 1/2; \text{ and } \mu\{f < m(f)\} \leq 1/2.$$

For a measurable set $Q \subset \Omega$, the median of f on Q is a number $m_Q(f)$ that is a median for $f\chi_Q : Q \rightarrow \mathbb{R}$, on the probability measure on Q defined by $\mu_Q(A) = \mu(A \cap Q)/\mu(Q)$. In other words, $m_Q(f)$ satisfies

$$\mu\{x \in Q : f(x) > m_Q(f)\} \leq \mu(Q)/2; \text{ and } \mu\{x \in Q : f(x) < m_Q(f)\} \leq \mu(Q)/2.$$

It is easy to convince oneself that this definition is equivalent to the perhaps more common definition of median (cf. [69, pag 134]) which requires that

$$\mu\{x \in Q : f(x) \geq m_Q(f)\} \geq \mu(Q)/2; \text{ and } \mu\{x \in Q : f(x) \leq m_Q(f)\} \geq \mu(Q)/2.$$

Recall the following basic property of medians (cf. [92], [69, page 134], etc.): with constants independent of Q , we have

$$\inf_c \frac{1}{\mu(Q)} \int_Q |f - c| d\mu \simeq \frac{1}{\mu(Q)} \int_Q |f - m_Q(f)| d\mu.$$

It follows that when dealing with BMO we can use medians rather than averages, and we have

$$\|f\|_{BMO} \simeq \sup_Q \frac{1}{\mu(Q)} \int_Q |f - m_Q(f)| d\mu,$$

where the sup is taken over all open balls contained in Ω . Our next remark is that the signed rearrangement can be used to compute medians.

THEOREM 26. $m_Q(f) = (f\chi_Q)_\mu^*(\mu(Q)/2)$.

PROOF. By definition

$$\mu\{f\chi_Q > (f\chi_Q)_\mu^*(\mu(Q)/2)\} \leq \mu(Q)/2.$$

Now

$$\mu\{f\chi_Q < (f\chi_Q)_\mu^*(\mu(Q)/2)\} = \mu\{-f\chi_Q > -(f\chi_Q)_\mu^*(\mu(Q)/2)\}$$

But since

$$(-f\chi_Q)_\mu^*(t) = -(f\chi_Q)_\mu^*(1-t)$$

it follows that

$$(-f\chi_Q)_\mu^*(\mu(Q)/2) = -(f\chi_Q)_\mu^*(\mu(Q)/2).$$

Consequently

$$\begin{aligned} \mu\{f\chi_Q < (f\chi_Q)_\mu^*(\mu(Q)/2)\} &= \mu\{-f\chi_Q > -(f\chi_Q)_\mu^*(\mu(Q)/2)\} \\ &= \mu\{-f\chi_Q > (-f\chi_Q)_\mu^*(\mu(Q)/2)\} \\ &\leq \mu(Q)/2 \text{ (by definition).} \end{aligned}$$

Therefore $(f\chi_Q)_\mu^*(\mu(Q)/2)$ is a median as we wished to show. \square

As a consequence we have the following John-Stromberg inequality: for any cube Q ,

$$(2.1) \quad (f_\mu^{**}(\mu(Q)/2) - f_\mu^*(\mu(Q)/2)) \leq \frac{1}{2} \|f\|_{BMO}.$$

3. Oscillation, BMO and K -functionals

As is well known in the Euclidean world or even for fairly general metric spaces (cf. [26]) one can realize BMO as a limiting Lip space. The easiest way to see this is to through the equivalence

$$\|f\|_{Lip_\alpha} \simeq \sup_Q \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f - f_Q| dx < \infty.$$

From this point of view BMO corresponds to a Lip space of order $\alpha = 0$.

This observation leads naturally to consider the analogs of the results of Chapter 4 in the context of BMO .

THEOREM 27. *Suppose that (Ω, d, μ) is a finite metric space such that there exists an absolute constant $C > 0$ such that for all $f \in L^1(\Omega)$ we have*

$$(3.1) \quad |f|_\mu^{**}(t) - |f|_\mu^*(t) \leq C \|f\|_*, \quad 0 < t < \mu(\Omega).$$

Then, for every r.i. space $X(\Omega)$ there exists a constant $c > 0$ such that

$$(3.2) \quad |f|_\mu^{**}(t) - |f|_\mu^*(t) \leq c \frac{K(\phi_X(t), f; X(\Omega), BMO(\Omega))}{\phi_X(t)}, \quad 0 < t < \mu(\Omega),$$

where

$$K(t, f; X(\Omega), BMO(\Omega)) = \inf_{h \in BMO} \{\|f - h\|_X + t \|h\|_*\}.$$

PROOF. The proof follows exactly the same lines as the proof of Theorem 7, so we shall be brief. We start by noting three important properties that $\|f\|_*$ shares with $\|\nabla f\|$: (i) For any constant c , $\|f + c\|_* = \|f\|_*$, (ii) $\|f\|_* \leq \|f\|_*$, and more generally (iii) for any Lip 1 function Ψ , $\|\Psi(f)\|_* \leq \|f\|_*$. As a first step we shall assume that $f \in L^\infty$. Let $t > 0$, then using the corresponding arguments⁶ in Theorem 7 shows that we have

$$(3.3) \quad \inf_{0 \leq h \in BMO} \{\|f - h\|_X + \phi_X(t) \|h\|_*\} \leq K(\phi_X(t), f; X(\Omega), BMO(\Omega)),$$

and

$$(3.4) \quad \inf_{0 \leq h \leq |f|, h \in BMO} \{\|f - h\|_X + \phi_X(t) \|h\|_*\} \leq 2 \inf_{0 \leq h \in BMO} \{\|f - h\|_X + \phi_X(t) \|h\|_*\}.$$

To prove (3.2) we proceed as in the proof of Theorem 7 until we arrive to

$$|f|_\mu^{**}(t) - |f|_\mu^*(t) \leq (|f - h|)_\mu^{**}(t) + |h|_\mu^{**}(t) - |h|_\mu^*(t),$$

⁶When comparing the arguments note that, in particular, $\|f\|_*$ behaves well for truncations: if $\Psi(x) = \min(x, c)$, then $\|\Psi(f)\|_* \leq \|f\|_*$.

for all $h \in BMO$ such that $0 \leq h \leq |f|$. Note that

$$\begin{aligned} (|f| - h)_\mu^{**}(t) &= \frac{1}{t} \int_0^t (|f| - h)_\mu^*(s) ds \\ &\leq \frac{\| |f| - h \|_X \phi_{X'}(t)}{t} \quad (\text{H\"older's inequality}) \\ &= \frac{\| |f| - h \|_X}{\phi_X(t)} \quad (\text{since } \phi_{X'}(t) \phi_X(t) = t). \end{aligned}$$

On the other hand, by (3.1)

$$|h|_\mu^{**}(t) - |h|_\mu^*(t) \leq C \|h\|_*.$$

Therefore, combining our findings we see that

$$\begin{aligned} |f|_\mu^{**}(t) - |f|_\mu^*(t) &\leq \inf_{0 \leq h \leq |f|} \left\{ \frac{\| |f| - h \|_X}{\phi_X(t)} + C \|h\|_* \right\} \\ &= \frac{c}{\phi_X(t)} \inf_{0 \leq h \leq |f|} \{ \| |f| - h \|_X + \phi_X(t) \|h\|_* \} \\ &\leq \frac{C}{\phi_X(t)} K(\phi_X(t), f; X(\Omega), BMO(\Omega)) \quad (\text{by (3.4) and (3.3)}). \end{aligned}$$

To lift the assumption that $f \in L^\infty$ we proceed once again as in the proof of Theorem 7. \square

CHAPTER 8

Estimation of growth “envelopes”

1. Summary

Triebel and his school, in particular we should mention here the extensive work of Haroske, have studied the concept of *envelopes* (cf. [48], a book mainly devoted to the computation of growth and continuity envelopes of function spaces defined on \mathbb{R}^n). On the other hand, as far as we are aware, the problem of estimating growth envelopes for Sobolev or Besov spaces based on general measure spaces has not been treated systematically in the literature. For a function space $Z(\Omega)$, which we should think as measuring smoothness, one attempts to find precise estimates of (*the growth envelope*)

$$E^Z(t) = \sup_{\|f\|_{Z(\Omega)} \leq 1} |f|^*(t).$$

A related problem is the estimation of *continuity envelopes* (cf. [48]). For example, suppose that $Z := Z(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, then we let

$$E_C^Z(t) = \sup_{\|f\|_Z \leq 1} \frac{\omega_{L^\infty}(t, f)}{t},$$

and the problem at hand is to obtain precise estimates of $E_C^Z(t)$.

In this chapter we estimate growth envelopes of function spaces based on metric probability spaces using our symmetrization inequalities. Most of the results we shall obtain, including those for Gaussian function spaces, are apparently new. Our method, moreover, gives a unified approach.

In a somewhat unrelated earlier work [63], we proposed some abstract ideas on how to study certain convergence and compactness properties in the context of interpolation scales. In Section 6 we shall briefly show a connection with the estimation of envelopes.

2. Spaces defined on measure spaces with Euclidean type profile

To fix ideas, and for easier comparison, in this section we consider metric probability¹ spaces (Ω, d, μ) such that the corresponding profiles satisfy

$$(2.1) \quad t^{1-1/n} \preceq I_\Omega(t), \quad t \in (0, 1/2).$$

¹Note that, when we are dealing with domains Ω with finite measure, we can usually assume without loss that we are dealing with functions such that $|f|^{**}(\infty) = 0$. Indeed, we have

$$|f|^{**}(t) \leq \frac{1}{t} \|f\|_{L^1(\Omega)}.$$

For the usual function spaces on \mathbb{R}^n , we can usually work with functions in $C_0(\mathbb{R}^n)$, which again obviously satisfy $|f|^{**}(\infty) = 0$.

Particular examples are the $\mathcal{J}_{1-\frac{1}{n}}$ -Maz’ya domains on \mathbb{R}^n . By the L  vy-Gromov isoperimetric inequality, Riemannian manifolds with positive Ricci curvature also satisfy (2.1).

In this context the basic rearrangement inequalities (cf. [(1.2), (1.9), Chapter 1]) take the following form, if $f \in Lip(\Omega)$, then

$$(2.2) \quad |f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t) \leq t^{1/n} |\nabla f|_{\mu}^{**}(t), \quad t \in (0, 1/2),$$

and, if $f \in X + S_X$, then

$$(2.3) \quad |f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t) \leq \frac{K(t^{1/n}, f, X, S_X)}{\phi_X(t)}, \quad t \in (0, 1/2).$$

THEOREM 28. *Let $X = X(\Omega)$ be a r.i. space on Ω , and let $\bar{S}_X(\Omega)$ be defined by*

$$\bar{S}_X(\Omega) = \left\{ f \in Lip(\Omega) : \|f\|_{\bar{S}_X(\Omega)} = \|\nabla f\|_X + \|f\|_X < \infty \right\}.$$

Then,

$$(2.4) \quad E^{\bar{S}_X(\Omega)}(t) \leq \int_t^1 s^{1/n-1} \phi_{\bar{X}'}(s) \frac{ds}{s}, \quad t \in (0, 1/2).$$

(ii) *In particular, if $X = L^p$, $1 \leq p < n$, then (compare with [48])*

$$(2.5) \quad E^{\bar{S}_{L^p}(\Omega)}(t) \leq t^{1/n-1/p}, \quad t \in (0, 1/2).$$

PROOF. Let f be such that $\|f\|_{\bar{S}_X(\Omega)} \leq 1$. Using the fundamental theorem of Calculus we can write

$$(2.6) \quad |f|_{\mu}^{**}(t) - |f|_{\mu}^{**}(1/2) = \int_t^{1/2} \left(|f|_{\mu}^{**}(s) - |f|_{\mu}^{*}(s) \right) \frac{ds}{s}.$$

This representation combined with (2.2) and H  lder’s inequality², yields

$$\begin{aligned} |f|_{\mu}^{**}(t) &\leq c_n \int_t^{1/2} s^{1/n} |\nabla f|_{\mu}^{**}(s) \frac{ds}{s} + |f|_{\mu}^{**}(1/2) \\ &\leq c_n \int_t^1 s^{1/n-1} \|\nabla f|_{\chi(0,s)}\|_{\bar{X}} \phi_{\bar{X}'}(s) \frac{ds}{s} + 2 \|f\|_{L^1} \\ &\leq \|f\|_{S_X} c_n \int_t^1 s^{1/n-1} \phi_{\bar{X}'}(s) \frac{ds}{s} + 2 \|f\|_{L^1} \\ &\leq c_n \int_t^1 s^{1/n-1} \phi_{\bar{X}'}(s) \frac{ds}{s} + 2c, \end{aligned}$$

²Write

$$\begin{aligned} s |\nabla f|_{\mu}^{**}(s) &= \int_0^s |\nabla f|^{*}(u) du \\ &\leq \|\nabla f|_{\chi(0,s)}\|_{\bar{X}} \|\chi(0,s)\|_{\bar{X}'} \\ &= \|\nabla f|_{\chi(0,s)}\|_{\bar{X}} \phi_{\bar{X}'}(s). \end{aligned}$$

where in the last step we used the fact that $\|f\|_{\bar{S}_X(\Omega)} \leq 1$, and $\|f\|_{L^1} \leq c\|f\|_X$. Therefore,

$$\begin{aligned} E^{\bar{S}_X(\Omega)}(t) &= \sup_{\|f\|_{\bar{S}_X(\Omega)} \leq 1} |f|_\mu^*(t) \\ &\leq \sup_{\|f\|_{\bar{S}_X(\Omega)} \leq 1} |f|_\mu^{**}(t) \\ &\leq c \left(\int_t^1 s^{1/n-1} \phi_{\bar{X}'}(s) \frac{ds}{s} + 1 \right), \quad t \in (0, 1/2). \end{aligned}$$

The second part of the result follows readily by computation since, if $X = L^p$, $1 \leq p < n$, then

$$\begin{aligned} \int_t^1 s^{1/n-1} \phi_{\bar{X}'}(s) \frac{ds}{s} &= \int_t^1 s^{1/n-1} s^{1-1/p} \frac{ds}{s} \\ &\leq \int_t^\infty s^{1/n-1} s^{1-1/p} \frac{ds}{s} \\ &\simeq t^{1/n-1/p}, \end{aligned}$$

and

$$1 \leq ct^{1/n-1/p}, \text{ for } t \in (0, 1/2).$$

□

REMARK 13. We can also deal in the same fashion with infinite measures. For comparison with [48] let us consider the case of \mathbb{R}^n provided with Lebesgue measure. In this case $I_{\mathbb{R}^n}(t) = c_n t^{1-1/n}$, for $t > 0$, and (2.2) is known to hold for all $t > 0$, and for all functions in $C_0(\mathbb{R}^n)$ (cf. [68]). For functions in $C_0(\mathbb{R}^n)$ we can replace (2.6) by

$$|f|^{**}(t) = \int_t^\infty (|f|^{**}(s) - |f|^*(s)) \frac{ds}{s}.$$

Suppose further that X is such that

$$\int_t^\infty s^{1/n-1} \phi_{\bar{X}'}(s) \frac{ds}{s} < \infty.$$

Then, proceeding with the argument given in the proof above, we see that there is no need to restrict the range of t 's for the validity of (2.4), (2.5), etc. Therefore, for $1 \leq p < n$, we have (compare with [48, Proposition 3.25])

$$(2.7) \quad E^{S_X^1(\mathbb{R}^n)}(t) \leq c \left(\int_t^\infty s^{1/n-1} \phi_{\bar{X}'}(s) \frac{ds}{s} + 1 \right), \quad t > 0.$$

The use of Hölder's inequality as effected in the previous theorem does not give the sharp result at the end point $p = n$. Indeed, following the previous method for $p = n$, we only obtain

$$\begin{aligned} E^{W_{L^n}^1(\Omega)}(t) &\preceq \int_t^1 s^{1/n-1} s^{1-1/n} \frac{ds}{s} \\ &\preceq \ln \frac{1}{t}, \quad t \in (0, 1/2). \end{aligned}$$

Our next result shows that using (2.2) in a slightly different form (applying Hölder's inequality on the left hand side) we can obtain the sharp estimate in the limiting cases (compare with [48, Proposition 3.27]).

THEOREM 29. $E^{\bar{S}_{L^n}(\Omega)}(t) \preceq \left(\ln \frac{1}{t}\right)^{1/n'}$, for $t \in (0, 1/2)$.

PROOF. Suppose that $\|f\|_{\bar{S}_{L^n}(\Omega)} \leq 1$. First we rewrite (2.2) as

$$\left(\frac{|f|_{\mu}^{**}(s) - |f|_{\mu}^*(s)}{s^{1/n}} \right)^n \leq c_n \left(|\nabla f|_{\mu}^{**}(s) \right)^n, s \in (0, 1/2).$$

Integrating, we thus find,

$$\begin{aligned} \left(\int_t^{1/2} \left(|f|_{\mu}^{**}(s) - |f|_{\mu}^*(s) \right)^n \frac{ds}{s} \right)^{1/n} &= \int_t^{1/2} \left(\frac{|f|_{\mu}^{**}(s) - |f|_{\mu}^*(s)}{s^{1/n}} \right)^n ds \\ &\leq c_n \int_t^1 \left(|\nabla f|_{\mu}^{**}(s) \right)^n ds \\ &\leq C_n \|\nabla f\|_{L^n}^n \text{ (by Hardy's inequality)} \\ &\leq C_n. \end{aligned}$$

Now, for $t \in (0, 1/2)$,

$$\begin{aligned} |f|_{\mu}^{**}(t) - |f|_{\mu}^{**}(1/2) &= \int_t^{1/2} \left(|f|_{\mu}^{**}(s) - |f|_{\mu}^*(s) \right) \frac{ds}{s} \\ &\leq \left(\int_t^{1/2} \left(|f|_{\mu}^{**}(s) - |f|_{\mu}^*(s) \right)^n \frac{ds}{s} \right)^{1/n} \left(\int_t^1 \frac{ds}{s} \right)^{1/n'} \text{ (Hölder's inequality)} \\ &\leq C_n^{1/n} \left(\ln \frac{1}{t} \right)^{1/n'}. \end{aligned}$$

Therefore,

$$\begin{aligned} |f|_{\mu}^{**}(t) &\leq C_n^{1/n} \left(\ln \frac{1}{t} \right)^{1/n'} + \|f\|_{L^1} \\ &\leq C_n^{1/n} \left(\ln \frac{1}{t} \right)^{1/n'} + \|f\|_{L^n} \\ &\leq C_n^{1/n} \left(\ln \frac{1}{t} \right)^{1/n'}, \text{ for } t \in (0, 1/2). \end{aligned}$$

Consequently,

$$(2.8) \quad E^{\bar{S}_{L^n}(\Omega)}(t) \preceq \left(\ln \frac{1}{t} \right)^{1/n'}, \text{ for } t \in (0, 1/2).$$

□

The case $p > n$, is somewhat less interesting for the computation of growth envelopes since we should have $E^{\bar{S}_{L^p}(\Omega)}(t) \leq c$. We now give a direct proof of this fact just to show that our method unifies all the cases.

PROPOSITION 4. Let $p > n$, then there exists a constant $c = c(n, p)$ such that

$$E^{\bar{S}_{L^p}(\Omega)}(t) \leq c, \text{ } t \in (0, 1/2).$$

PROOF. Suppose that $\|f\|_{\bar{S}_{L^p}(\Omega)} \leq 1$, and let $s \in (0, 1/2)$. We estimate as follows

$$\begin{aligned} |f|_{\mu}^{**}(s) - |f|_{\mu}^*(s) &\leq cs^{1/n-1} \int_0^s |\nabla f|_{\mu}^*(s) ds \\ &\leq cs^{1/n-1} \left(\int_0^1 |\nabla f|_{\mu}^*(s)^p ds \right)^{1/p} s^{1/p'} \\ &\leq cs^{1/n-1/p}. \end{aligned}$$

Thus, using a familiar argument, we see that for $t \in (0, 1/2)$,

$$\begin{aligned} |f|_{\mu}^{**}(t) - |f|_{\mu}^{**}(1/2) &\leq c \int_t^1 s^{1/n-1/p-1} ds \\ &\leq c \frac{1 - t^{1/n-1/p}}{(1/n - 1/p)} \\ &\leq \frac{c}{(1/n - 1/p)}. \end{aligned}$$

It follows that

$$\begin{aligned} |f|_{\mu}^*(t) &\leq |f|_{\mu}^{**}(t) \leq \frac{c}{(1/n - 1/p)} + |f|_{\mu}^{**}(1/2) \\ &\leq \frac{c}{(1/n - 1/p)}, \end{aligned}$$

and we obtain

$$E^{\bar{S}_{L^p}(\Omega)}(t) \leq c, \quad t \in (0, 1/2).$$

□

The methods discussed above apply to Sobolev spaces based on general r.i. spaces. As another illustration we now consider in detail the case of the Sobolev spaces based on the Lorentz spaces $L^{n,q}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded Lip domain of measure 1. The interest here lies in the fact that in the critical case $p = n$, the second index q plays an important role. Indeed, for $q = 1$, as is well known, we have $W_{L^{n,1}}^1 \subset L^\infty$, while this is no longer true for $W_{L^{n,q}}^1$ if $q > 1$. In particular, for the space $W_{L^{n,n}}^1 = W_{L^n}^1$. The next result thus extends Theorem 29 and provides an explanation of the situation we have just described through the use of growth envelopes.

THEOREM 30. *Let $1 \leq q \leq \infty$, then $E^{W_{L^{n,q}}^1(\Omega)}(t) \preceq (\ln \frac{1}{t})^{1/q'}$, for $t \in (0, 1/2)$.*

PROOF. Consider first the case $1 \leq q < \infty$. Suppose that $\|f\|_{W_{L^{n,q}}^1(\Omega)} \leq 1$. From (2.2) we get

$$(|f|^{**}(s) - |f|^*(s))^q \leq c_n \left(|\nabla f|^{**}(s) s^{1/n} \right)^q, \quad s \in (0, 1/2).$$

Then,

$$\begin{aligned}
|f|_{\mu}^{**}(t) - |f|_{\mu}^{**}(1/2) &= \int_t^{1/2} \left(|f|_{\mu}^{**}(s) - |f|_{\mu}^{*}(s) \right) \frac{ds}{s} \\
&\leq \left(\int_t^{1/2} \left(|f|_{\mu}^{**}(s) - |f|_{\mu}^{*}(s) \right)^q \frac{ds}{s} \right)^{1/q} \left(\int_t^{1/2} \frac{ds}{s} \right)^{1/q'} \\
&\leq c \left(\int_t^{1/2} \left(|\nabla f|_{\mu}^{**}(s) s^{1/n} \right)^q \frac{ds}{s} \right)^{1/q} \left(\int_t^{1/2} \frac{ds}{s} \right)^{1/q'} \\
&\leq c \|\nabla f\|_{L^{n,q}} \left(\ln \frac{1}{t} \right)^{1/q'}.
\end{aligned}$$

Therefore, as before

$$\begin{aligned}
|f|_{\mu}^{**}(t) &\leq c \|\nabla f\|_{L^{n,q}} \left(\ln \frac{1}{t} \right)^{1/q'} + |f|_{\mu}^{**}(1/2) \\
&\leq c \left(\ln \frac{1}{t} \right)^{1/q'}, \quad t \in (0, 1/2),
\end{aligned}$$

and the desired estimate for $E^{W_{L^{n,q}}^1(\Omega)}(t)$ follows.

When $q = \infty$, and $\|f\|_{W_{L^{n,\infty}}^1(\Omega)} \leq 1$, we estimate

$$\begin{aligned}
|f|_{\mu}^{**}(s) - |f|_{\mu}^{*}(s) &\leq c_n |\nabla f|^{**}(s) s^{1/n} \\
&\leq c_n \|\nabla f\|_{L(n,\infty)} \\
&\leq c_n.
\end{aligned}$$

Consequently,

$$|f|_{\mu}^{**}(t) - |f|_{\mu}^{**}(1/2) \leq c_n \int_t^1 \frac{ds}{s}$$

and we readily get

$$E^{W_{L^{n,\infty}}^1(\Omega)}(t) \leq c \left(\ln \frac{1}{t} \right), \quad t \in (0, 1/2).$$

□

REMARK 14. *Note that, in particular,*

$$E^{W_{L^{n,1}}^1(\Omega)}(t) \leq c,$$

which again reflects the fact that $W_{L^{n,1}}^1(\Omega) \subset L^{\infty}(\Omega)$.

REMARK 15. *As before, all the previous results hold for the $W_{L^{p,q}}^1(\mathbb{R}^n)$ spaces.*

We now show that a similar method, replacing the use of (2.2) by (2.3), allow us to obtain sharp estimates for growth envelopes of Besov spaces (see [Chapter 7, (1.6)].

THEOREM 31. *Let $p > n > 1$, $1 \leq q \leq \infty$. Then*

$$E^{B_{p,q}^{n/p}([0,1]^n)}(t) \preceq \left(\log \frac{1}{t} \right)^{1/q'}, \quad t \in (0, 1/2).$$

PROOF. The starting point is

$$|f|_{\mu}^{**}(t) - |f|_{\mu}^{*}(t) \leq c \frac{\omega_{L^p}(t^{1/n}, f)}{t^{1/p}}, \quad t \in (0, 1/2).$$

Then,

$$\begin{aligned} |f|_{\mu}^{**}(t) - |f|_{\mu}^{**}(1/2) &= \int_t^{1/2} \left(|f|_{\mu}^{**}(s) - |f|_{\mu}^{*}(s) \right) \frac{ds}{s} \\ &\leq c \int_t^{1/2} \frac{\omega_{L^p}(s^{1/n}, f)}{s^{1/p}} \frac{ds}{s} \\ &\leq c \left(\int_t^1 \left(\frac{\omega_{L^p}(s^{1/n}, f)}{s^{1/p}} \right)^q \frac{ds}{s} \right)^{1/q} \left(\int_t^1 \frac{ds}{s} \right)^{1/q'} \quad (\text{H\"older's inequality}) \\ &\leq c \|f\|_{B_{p,q}^{n/p}([0,1]^n)} \left(\log \frac{1}{t} \right)^{1/q'}. \end{aligned}$$

Thus, a familiar argument now gives (compare with [48, (1.9)])

$$E^{B_{p,q}^{n/p}([0,1]^n)}(t) \preceq \left(\log \frac{1}{t} \right)^{1/q'}, \quad t \in (0, 1/2).$$

□

3. Continuity Envelopes

Suppose that $Z := Z(\mathbb{R}^n) \subset C(\mathbb{R}^n)$, then (cf. [48] and the references therein) one defines the continuity envelope by

$$E_C^Z(t) = \sup_{\|f\|_{Z(\mathbb{R}^n)} \leq 1} \frac{\omega_{L^\infty}(t, f)}{t}.$$

At this point it is instructive to recall some known interpolation inequalities. Let $\|f\|_{\dot{W}_{L^{n,1}}^1} = \int_0^\infty |\nabla f|^*(s) s^{1/n} \frac{ds}{s}$. We interpolate the following known embeddings (cf. [87]): for $f \in C_0^\infty(\mathbb{R}^n)$, we have

$$\|f\|_{\dot{W}_{L^\infty}^1(\mathbb{R}^n)} \preceq \|f\|_{\dot{W}_{L^\infty}^1(\mathbb{R}^n)}, \quad \text{and} \quad \|f\|_{L^\infty(\mathbb{R}^n)} \preceq \|f\|_{\dot{W}_{L^{n,1}}^1(\mathbb{R}^n)}.$$

Consequently

$$(3.1) \quad K(t, f; L^\infty(\mathbb{R}^n), \dot{W}_{L^\infty}^1(\mathbb{R}^n)) \preceq K(t, f, \dot{W}_{L^{n,1}}^1(\mathbb{R}^n), \dot{W}_{L^\infty}^1(\mathbb{R}^n)).$$

It is well that for continuous functions we have (cf. [13])

$$K(t, f; L^\infty(\mathbb{R}^n), \dot{W}_{L^\infty}^1(\mathbb{R}^n)) = \omega_{L^\infty}(t, f) \simeq \sup_{|h| \leq t} \|f(x+h) - f(x)\|_{L^\infty}.$$

On the other hand using [65, Theorem 2] and Holmstedt's Lemma (see [14, Theorem 3.6.1]) we find

$$K(t, f, \dot{W}_{L^{n,1}}^1(\mathbb{R}^n), \dot{W}_{L^\infty}^1(\mathbb{R}^n)) \simeq \int_0^{t^n} |\nabla f|^*(s) s^{1/n} \frac{ds}{s}.$$

Inserting this information back to (3.1) we find

$$\omega_{L^\infty}(t, f) \preceq \int_0^{t^n} |\nabla f|^*(s) s^{1/n} \frac{ds}{s}.$$

Therefore,

$$\begin{aligned} \frac{\omega_{L^\infty}(t, f)}{t} &\preceq \frac{1}{t} \int_0^{t^n} |\nabla f|^*(s) s^{1/n} \frac{ds}{s} \\ &\preceq \frac{1}{t} \|\nabla f\|_{L^{p,q}} \left(\int_0^{t^n} s^{(1/n-1/p)q'} \frac{ds}{s} \right)^{1/q'} \\ &\preceq \frac{1}{t} \|\nabla f\|_{L^{p,q}} t^{1-n/p}. \end{aligned}$$

Thus, we have (compare with [48, (1.15)]) that for $1 \leq q < \infty$,

$$(3.2) \quad E_C^{W_{L^{p,q}}^1(\mathbb{R}^n)}(t) \preceq t^{-n/p}.$$

REMARK 16. *It is actually fairly straightforward at this point to derive a general relation between E^Z and E_C^Z . In [64] we have shown for $f \in C_0^\infty(\mathbb{R}^n)$ we have*

$$(3.3) \quad |f|^{**}(t) - |f|^*(t) \leq c_n \omega_{L^\infty}(t^{1/n}, f), \quad t > 0.$$

We now proceed formally, although the details can be easily filled-in by the interested reader. From (3.3) we find

$$\frac{|f|^{**}(t) - |f|^*(t)}{t} \leq c_n \frac{\omega_{L^\infty}(t^{1/n}, f)}{t}, \quad t > 0.$$

Then

$$\begin{aligned} |f|^*(t) &\leq |f|^{**}(t) \\ &= \int_t^\infty \frac{|f|^{**}(s) - |f|^*(s)}{s} ds \quad (\text{since } f^{**}(\infty) = 0) \\ &\leq c_n \int_t^\infty \frac{\omega_{L^\infty}(s^{1/n}, f)}{s} ds. \end{aligned}$$

Taking supremum over the unit ball of $Z(\mathbb{R}^n)$ we obtain

$$E^{Z(\mathbb{R}^n)}(t) \preceq \int_{t^{1/n}}^\infty E_C^{Z(\mathbb{R}^n)}(s) ds.$$

Thus, for example, from (3.2) we find that for $p < n$

$$\begin{aligned} E^{W_{L^{p,q}}^1(\mathbb{R}^n)}(t) &\preceq \int_{t^{1/n}}^\infty s^{-n/p} ds \\ &\simeq t^{1/n-1/p}, \end{aligned}$$

which should be compared with (2.5).

4. General isoperimetric profiles

In the previous sections we have focussed mainly on function spaces on domains with isoperimetric profiles of Euclidean type; but our inequalities also provide a unified setting to study estimates for general profiles. For a metric measure space (Ω, d, μ) of finite measure we consider r.i. spaces $X(\Omega)$. Let $0 < \theta < 1$ and $1 \leq q \leq \infty$, the homogeneous Besov space $\dot{b}_{X,q}^\theta(\Omega)$ is defined by

$$\dot{b}_{X,q}^\theta(\Omega) = \left\{ f \in X + S_X : \|f\|_{\dot{b}_{X,q}^\theta(\Omega)} = \left(\int_0^{\mu(\Omega)} (K(s, f; X, S_X) s^{-\theta})^q \frac{ds}{s} \right)^{1/q} < \infty \right\},$$

with the usual modifications when $q = \infty$. The Besov space $b_{X,q}^\theta(\Omega, \mu)$ is defined by

$$\|f\|_{b_{X,q}^\theta(\Omega, \mu)} = \|f\|_X + \|f\|_{\dot{b}_{X,q}^\theta(\Omega)}.$$

Notice that if $X = L^p$, then $\dot{b}_{L^p,q}^\theta(\Omega) = \dot{b}_{p,q}^\theta(\Omega)$ (resp. $b_{L^p,q}^\theta(\Omega) = b_{p,q}^\theta(\Omega)$) [see Chapter 7, (1.1)].

THEOREM 32. *Let X be a r.i. space on Ω . Let $g(s) = \frac{s}{I_\Omega(s)}$ where I_Ω denotes the isoperimetric profile of (Ω, d, μ) . Let $b_{X,q}^\theta(\Omega)$ be a Besov space ($0 < \theta < 1$, $1 < q < \infty$), then for $t \in (0, \mu(\Omega)/2)$ we have that*

$$E^{b_{X,q}^\theta(\Omega)}(t) \leq c \left(1 + \left(\int_t^{\mu(\Omega)/2} \left(g(s)^\theta \left(\frac{g(s)}{g'(s)} \right)^{1/q} \right)^{q'} \frac{ds}{(s\phi_X(s))^{q'}} \right) \right),$$

where, as usual $1/q' + 1/q = 1$.

PROOF. Let $f \in X + S_X$, and let us write $K(t, f; X, S_X) := K(t, f)$. By Theorem 7 we know that

$$|f|_\mu^{**}(t) - |f|_\mu^*(t) \leq c \frac{K(g(t), f)}{\phi_X(t)}, \quad 0 < t < \mu(\Omega).$$

Taking in account that, $(-|f|_\mu^{**})'(t) = (|f|_\mu^{**}(t) - |f|_\mu^*(t))/t$, we get

$$|f|_\mu^{**}(t) - |f|_\mu^{**}(\mu(\Omega)/2) = \int_t^{\mu(\Omega)/2} (-|f|_\mu^{**})'(s) ds \leq c \int_t^{\mu(\Omega)/2} \frac{K(g(s), f)}{\phi_X(s)} \frac{ds}{s}.$$

Since $I_\Omega(s)$ is a concave continuous increasing function on $(0, \mu(\Omega)/2)$, $g(s)$ is differentiable on $(0, \mu(\Omega)/2)$. Then, by Hölder's inequality we have

$$\begin{aligned} R(t) &= \int_t^{\mu(\Omega)/2} \frac{K(g(s), f)}{\phi_X(s)} \frac{ds}{s} \\ &= \int_t^{\mu(\Omega)/2} K(g(s), f) \left(\frac{g(s)}{g'(s)} \right)^\theta \left(\frac{g'(s)}{g(s)} \right)^{1/q} \left(\frac{g(s)}{g'(s)} \right)^{1/q} \frac{ds}{s\phi_X(s)} \\ &\leq R_1(t)R_2(t), \end{aligned}$$

where

$$R_1(t) = \left(\int_t^{\mu(\Omega)/2} (K(g(s), f) g(s)^{-\theta})^q \left(\frac{g'(s)}{g(s)} \right) ds \right)^{1/q},$$

and

$$(4.1) \quad R_2(t) = \left(\int_t^{\mu(\Omega)/2} \left(g(s)^\theta \left(\frac{g(s)}{g'(s)} \right)^{1/q} \right)^{q'} \frac{ds}{(s\phi_X(s))^{q'}} \right)^{1/q'}.$$

By a change of variables

$$(4.2) \quad R_1(t) = \left(\int_{g^{-1}(t)}^{g^{-1}(\mu(\Omega)/2)} (K(z, f) z^{-\theta})^q \frac{dz}{z} \right)^{1/q} \leq \|f\|_{\dot{b}_{X,q}^\theta(\Omega)}$$

Combining (4.1) and (4.2) we obtain

$$\begin{aligned} |f|_{\mu}^{**}(t) &\leq c \|f\|_{b_{X,q}^{\theta}(\Omega)} R_2(t) + 2 \|f\|_X \\ &\leq 2c(1 + R_2(t)) \|f\|_{b_{X,q}^{\theta}(\Omega)}. \end{aligned}$$

Therefore, taking sup over all f such that $\|f\|_{b_{X,q}^{\theta}(\Omega)} \leq 1$ we see that

$$E^{b_{X,q}^{\theta}(\Omega)}(t) \leq 2c(1 + R_2(t)), \quad t \in (0, \mu(\Omega)/2).$$

□

EXAMPLE 2. Consider the Gaussian measure (\mathbb{R}^n, γ_n) . Then (cf. [16]) we can take as isoperimetric estimator

$$I_{\gamma_n}(t) = t \left(\log \frac{1}{t} \right)^{1/2}, \quad t \in (0, 1/2).$$

Thus,

$$g(t) = \frac{1}{(\log \frac{1}{t})^{1/2}} \quad \text{and} \quad g'(s) = \frac{1}{2 (\log \frac{1}{s})^{\frac{3}{2}} s},$$

and

$$\begin{aligned} &\left(1 + \left(\int_t^{1/2} \left(g(s)^{\theta} \left(\frac{g(s)}{g'(s)} \right)^{1/q} \right)^{q'} \frac{ds}{(s\phi_X(s))^{q'}} \right)^{1/q'} \right) \\ &= \left(\int_t^{1/2} \left(\log \frac{1}{s} \right)^{q'(1-\frac{\theta}{2})-1} \frac{ds}{s (\phi_X(s))^{q'}} \right)^{1/q'} \\ &\leq \frac{1}{\phi_X(t)} \left(\int_t^{1/2} \left(\log \frac{1}{s} \right)^{q'(1-\frac{\theta}{2})-1} \frac{ds}{s} \right)^{1/q'} \\ &\preceq \frac{1}{\phi_X(t)} \left(\log \frac{1}{t} \right)^{(1-\frac{\theta}{2})}. \end{aligned}$$

Therefore we find that

$$E^{B_{X,q}^{\theta}(\mathbb{R}^n, \gamma_n)}(t) \preceq \frac{1}{\phi_X(t)} \left(\log \frac{1}{t} \right)^{(1-\frac{\theta}{2})}, \quad t \in (0, 1/2).$$

5. Envelopes for higher order spaces

In general it is not clear how to define higher order Sobolev and Besov spaces in metric spaces. On the other hand for classical domains (Euclidean, Riemannian manifolds, etc.) there is a well developed theory of embeddings that one can use to estimate growth envelopes. The underlying general principle is very simple. Suppose that the function space $Z = Z(\Omega)$ is continuously embedded in $Y = Y(\Omega)$ and Y is a rearrangement invariant space, then, since (cf. 2.4), Chapter 2) $Y \subset M(Y)$, where $M(Y)$ is the Marcinkiewicz space associated with Y (cf. Section 2 in Chapter 2), we have (cf. (2.3), Chapter 2) for all $f \in Z$,

$$\sup_t |f|^{*}(t) \phi_Y(t) \leq \|f\|_Y \leq c \|f\|_Z,$$

where $\phi_Y(t)$ is the fundamental function of Y , and c is the norm of the embedding $Z \subset Y$. Consequently, for all $f \in Z$, with $\|f\|_Z \leq 1$, for all $t > 0$,

$$|f|^*(t) \leq \frac{c}{\phi_Y(t)}.$$

Therefore,

$$E^Z(t) \preceq \frac{1}{\phi_Y(t)}.$$

For example, suppose that $p < \frac{n}{k}$, then from

$$W_p^k(\mathbb{R}^n) \subset L^{q,p}, \text{ with } \frac{1}{q} = \frac{1}{p} - \frac{k}{n}$$

and

$$\phi_{L^{q,p}}(t) = t^{1/q} = t^{1/p-k/n}$$

we get (compare with (2.5) above and [48, (1.7)])

$$E^{W_p^k(\mathbb{R}^n)}(t) \preceq t^{k/n-1/p}.$$

In the limiting case we have (cf. [9], [77])

$$W_{\frac{n}{k}}^k(\mathbb{R}^n) \subset L^{[\infty, \frac{n}{k}]}.$$

For comparison consider $W_{\frac{n}{k}}^k(\Omega)$, where Ω is a domain on \mathbb{R}^n , with $|\Omega| = 1$. One can readily estimate the decay of functions in $L^{[\infty, \frac{n}{k}]}$ as follows:

$$\begin{aligned} |f|^{**}(t) - |f|^{**}(1) &= \int_t^1 (|f|^{**}(s) - |f|^*(s)) \frac{ds}{s} \\ &\leq \left(\int_t^1 (|f|^{**}(s) - |f|^*(s))^{\frac{n}{k}} \frac{ds}{s} \right)^{1/q} \left(\int_t^1 \frac{ds}{s} \right)^{1/(\frac{n}{k})'} \\ &\leq \|f\|_{L^{[\infty, q]}} \left(\log \frac{1}{t} \right)^{1-\frac{k}{n}}. \end{aligned}$$

Combining these observations we see that for functions in the unit ball of $W_{\frac{n}{k}}^k(\Omega)$ we have

$$|f|^{**}(t) \preceq c \left(\log \frac{1}{t} \right)^{1-\frac{k}{n}}, \text{ for } t \in (0, 1/2).$$

Consequently

$$E^{W_{\frac{n}{k}}^k(\Omega)}(t) \preceq \left(\log \frac{1}{t} \right)^{1-\frac{k}{n}}.$$

In particular, when $k = 1$ then $1 - \frac{k}{n} = \frac{1}{n'}$, and the result coincides with Theorem 29 above.

Likewise we can deal with the case of general isoperimetric profiles but we shall leave the discussion for another occasion.

6. K and E functionals for families

It is of interest to point out a connection between the different “envelopes” discussed above and a more general concept introduced somewhat earlier in [63], in a different context. One of the tools introduced in [63] was to consider the K and E functionals for families, rather than single elements.

Given a compatible pair of spaces (X, Y) (cf. [14]), and a family of elements, $F \subset X + Y$, we can define the K -functional and E -functional³ of the family F by (cf. [63])

$$\begin{aligned} K(t, F; X, Y) &= \sup_{f \in F} K(t, f; X, Y). \\ E(t, F; X, Y) &= \sup_{f \in F} E(t, f; X, Y). \end{aligned}$$

The connection with the Triebel-Haroske envelopes can be seen from the following known computations. If we let $\|f\|_{L^0} = \mu\{\text{supp } f\}$, then

$$|f|^*(t) = E(t, f; L^0, L^\infty).$$

Therefore,

$$E^{Z(\Omega)}(t) = E(t, \text{unit ball of } Z(\Omega); L^0(\Omega), L^\infty(\Omega)).$$

Moreover, since on Euclidean space we have

$$\omega_{L^\infty}(t, f) \simeq K(t, f; L^\infty(\mathbb{R}^n), \dot{W}_{L^\infty}^1(\mathbb{R}^n))$$

we therefore see that

$$E_C^{Z(\mathbb{R}^n)}(t) = K(t, f; \text{unit ball of } Z(\mathbb{R}^n); L^\infty(\mathbb{R}^n), \dot{W}_{L^\infty}^1(\mathbb{R}^n)).$$

This suggests the general definition for metric spaces

$$E_C^{Z(\Omega)}(t) = K(t, f; \text{unit ball of } Z(\Omega); L^\infty(\Omega), S_{L^\infty}(\Omega)).$$

This provides a method to expand the known results to the metric setting using the methods discussed in this paper. Another interesting aspect of the connection we have established here lies in the fact, established in [63], that one can reformulate classical convergence and compactness criteria for function spaces (e.g. Kolmogorov’s compactness criteria for sets contained in L^p) in terms of conditions on these (new) functionals. For example, according to the Kolmogorov criteria, for a set of functions F to be compact on $L^p(\mathbb{R}^n)$ one needs the uniform continuity on F at zero of $\omega_{L^p}(t, F)$. In our formulation we replace this condition by demanding the continuity at zero of

$$K(t, F, L^p, W_{L^p}^1).$$

Again, to develop this material in detail is a long paper on its own, however, let us note in passing that the failure of compactness of the embedding $W_{L^p}^1(\Omega) \subset L^{\bar{p}}(\Omega)$, for $p = n$, is consistent with the blow up at zero predicted by the fact that the converse of (2.8) also holds. One should compare this with the estimate (2.7) which is consistent with the Relich compactness criteria for Sobolev embeddings on bounded domains, when $p < n$.

³Recall that (cf. [14], [63]),

$$\begin{aligned} K(t, f; X, Y) &= \inf\{\|f - g\|_X + t\|g\|_Y : g \in Y\} \\ E(t, f; X, Y) &= \inf\{\|f - g\|_Y : \|g\|_X \leq t\}. \end{aligned}$$

CHAPTER 9

Lorentz spaces with negative indices

1. Introduction and Summary

As we have shown elsewhere (cf. [9], [66]), the basic Euclidean inequality

$$f^{**}(t) - f^*(t) \leq c_n t^{1/n} |\nabla f|^{**}(t)$$

leads to the optimal Sobolev inequality

$$(1.1) \quad \|f\|_{L^{[\bar{p}, p]}} = \left\{ \int_0^\infty \left((|f|^{**}(t) - |f|^*(t)) t^{1/\bar{p}} \right)^p \frac{dt}{t} \right\}^{1/p} \leq c_n \|\nabla f\|_{L^p},$$

where $1 < p \leq n$, $\frac{1}{\bar{p}} = \frac{1}{p} - \frac{1}{n}$. The use of the $L^{[\bar{p}, p]}$ conditions makes it possible to consider the limiting case $p = n$ in a unified way. Now (1.1) is also meaningful when $p > n$, albeit the only reason for the restriction $p \leq n$, is that, if we don't impose it, then $\bar{p} < 0$, and thus the condition defined by $\|f\|_{L^{[\bar{p}, p]}} < \infty$ is not well understood. It is was shown in [77] that these conditions are meaningful. In this chapter we show a connection between the Lorentz $L^{[\bar{p}, p]}$ spaces with negative indices and Morrey's theorem.

1.1. Lorentz conditions. Let (Ω, d, μ) be a metric measure space. Let $0 < q \leq \infty$, $s \in \mathbb{R}$. We define

$$L^{[s, q]} = L^{[s, q]}(0, \mu(\Omega)) = \left\{ f \in L^1(\Omega) : \left\{ \int_0^{\mu(\Omega)} \left((|f|_\mu^{**}(t) - |f|_\mu^*(t)) t^{1/s} \right)^q \frac{dt}{t} \right\}^{1/q} < \infty \right\}.$$

For $0 < q \leq \infty$, $s \in [1, \infty]$, these spaces were defined in Chapter 7. They coincide with the usual $L^{s, q}$ spaces when $0 < q \leq \infty$, $s \in [1, \infty)$ (cf. [70]).

Our first observation is that $L^{[s, q]} \neq \emptyset$. Indeed, for $s < 0$, we have

$$\begin{aligned} 0 < \|\chi_A\|_{L^{[s, q]}} &= \frac{\mu(A)}{(q - q/s)^{1/q}} [\mu(A)^{q/s - q} - 1]^{1/q} \\ &\leq \frac{\mu(A)^{1/s}}{(q - q/s)^{1/q}}. \end{aligned}$$

It is important to remark that the cancellation at zero afforded by $|f|_\mu^{**}(t) - |f|_\mu^*(t)$ is crucial here. Indeed, if we attempt to extend the usual definition of Lorentz spaces by letting $s < 0$, then we find that $\|\chi_A\|_{L^{(s, q)}} = \int_0^{\mu(A)} t^{q/s} \frac{dt}{t} < \infty$ iff $\mu(A) = 0$.

2. The role of the $L^{[\bar{p}, p]}$ spaces in Morrey's theorem

For definiteness we work on \mathbb{R}^n with Lebesgue measure m . We show that many arguments we have discussed in this paper are available in the context of Lorentz spaces with negative index.

Let $f \in L^{[\bar{p}, p]}$ where $\frac{1}{\bar{p}} = \frac{1}{p} - \frac{1}{n} < 0$. Then, for $0 < t_1 < t_2$, we can write

$$\begin{aligned} f^{**}(t_1) - f^{**}(t_2) &= \int_{t_1}^{t_2} (f^{**}(t) - f^*(t)) t^{1/\bar{p}} t^{-1/\bar{p}} \frac{dt}{t} \\ &\leq \left(\int_{t_1}^{t_2} \left((f^{**}(t) - f^*(t)) t^{1/\bar{p}} \right)^p \frac{dt}{t} \right)^{1/p} \left(\int_{t_1}^{t_2} t^{-p'/\bar{p}} \frac{dt}{t} \right)^{1/p'} \\ &= \|f\|_{L^{[\bar{p}, p]}} \left(\int_{t_1}^{t_2} t^{-p'/\bar{p}} \frac{dt}{t} \right)^{1/p'}. \end{aligned}$$

Note that since $\frac{-p'}{\bar{p}} - 1 = \frac{p}{p-1} \left(\frac{n-p}{np} \right) - 1 = \frac{1}{p-1} \left[\frac{n-p-n}{n} \right] < 0$, the function $t^{-p'/\bar{p}-1}$ is decreasing and therefore,

$$\int_{t_1}^{t_2} t^{-p'/\bar{p}} \frac{dt}{t} \leq \int_0^{t_2-t_1} t^{-p'/\bar{p}} \frac{dt}{t} = \frac{-\bar{p}}{p'} |t_2 - t_1|^{\frac{-p'}{\bar{p}}}.$$

Thus,

$$(2.1) \quad f^{**}(t_1) - f^{**}(t_2) \leq \left(\frac{-\bar{p}}{p'} \right)^{1/p'} \|f\|_{L^{[\bar{p}, p]}} |t_2 - t_1|^{\frac{-1}{\bar{p}}}.$$

The localization property in this context takes the following form. Suppose that $f \in L^{[\bar{p}, p]}$ is such that there exists a constant $C > 0$, such that $\forall B$ open ball, it follows that $f\chi_B \in L^{[\bar{p}, p]}(0, m(B))$, with $\|f\chi_B\|_{L^{[\bar{p}, p]}} \leq C \|f\|_{L^{[\bar{p}, p]}}$. Then, from (2.1) we get

$$(f\chi_B)^{**}(t_1) - (f\chi_B)^{**}(t_2) \leq C \left(\frac{-\bar{p}}{p'} \right)^{1/p'} \|f\|_{L^{[\bar{p}, p]}} |t_2 - t_1|^{\frac{\alpha}{n}},$$

where $\alpha = 1 - \frac{n}{p}$. Applying this inequality replacing t_i by $t_i m(B)$, $i = 1, 2$; we get

$$(f\chi_B)^{**}(t_1) - (f\chi_B)^{**}(t_2) \leq C \left(\frac{-\bar{p}}{p'} \right)^{1/p'} \|f\|_{L^{[\bar{p}, p]}} |t_2 - t_1|^{\frac{\alpha}{n}} m(B)^{\frac{\alpha}{n}}.$$

Letting $t_1 \rightarrow 0, t_2 \rightarrow 1$, we then find

$$\operatorname{ess\,sup}_B(f) - \frac{1}{m(B)} \int_B f \leq C \left(\frac{-\bar{p}}{p'} \right)^{1/p'} \|f\|_{L^{[\bar{p}, p]}} m(B)^{a/n}.$$

Applying this inequality to $-f$ and adding we arrive at

$$\operatorname{ess\,sup}_B(f) - \operatorname{ess\,inf}_B f \leq 2C \left(\frac{-\bar{p}}{p'} \right)^{1/p'} \|f\|_{L^{[\bar{p}, p]}} m(B)^{a/n}.$$

Let $x, y \in \mathbb{R}^n$, and consider $B = B(x, 3|x-y|)$ (i.e. the ball centered at x , with radius $3|x-y|$), then

$$\begin{aligned} |f(x) - f(y)| &\leq \operatorname{ess\,sup}_B f - \operatorname{ess\,inf}_B f \\ &\leq c_n 2 \left(\frac{-\bar{p}}{p'} \right)^{1/p'} C \|f\|_{L^{[\bar{p}, p]}} |x - y|^\alpha. \end{aligned}$$

At this point we could appeal to (1.1) to conclude that

$$|f(x) - f(y)| \leq c_n 2 \left(\frac{-\bar{p}}{p'} \right)^{1/p'} C \|\nabla f\|_p |x - y|^\alpha.$$

Similar arguments apply when dealing with Besov spaces. In this case the point of departure is the corresponding replacement for (1.1) that is provided by the Besov embedding

$$\int \left[\left(|f|_{\mu}^{**}(t) - |f|_{\mu}^*(t) \right) t^{\frac{1}{p} - \frac{\theta}{n}} \right]^q \frac{dt}{t} \leq c \int \left[t^{-\frac{\theta}{n}} K(t^{1/n}, f; L^p, \dot{W}_{L^p}^1) \right]^q \frac{dt}{t},$$

where $\frac{1}{\bar{p}} = \frac{1}{p} - \frac{\theta}{n}$. $\theta \in (0, 1)$, $1 \leq q \leq \infty$. Notice that we don't assume anymore that $\theta p \leq n$.

REMARK 17. *In the usual argument the use of the Lorentz spaces with negative indices was implicit. The idea being that we can estimate $\left(\int_{t_1}^{t_2} ((f^{**}(t) - f^*(t)) t^{1/\bar{p}})^p \frac{dt}{t} \right)^{1/p}$ through the use of*

$$f^{**}(t) - f^*(t) \leq c_n t^{1/n} |\nabla f|^{**}(t).$$

Namely,

$$\begin{aligned} f^{**}(t_1) - f^{**}(t_2) &= \int_{t_1}^{t_2} (f^{**}(t) - f^*(t)) \frac{dt}{t} \\ &\leq \int_{t_1}^{t_2} t^{1/n-1} |\nabla f|^{**}(t) dt \quad (\text{basic inequality}) \\ &\leq \left(\int_{t_1}^{t_2} |\nabla f|^{**}(t)^p dt \right)^{1/p} \left(\int_{t_1}^{t_2} t^{p'(1/n-1)} dt \right)^{1/p'} \quad (\text{H\"older's inequality}) \\ &\leq c_p \|\nabla f\|_p \left(\int_{t_1}^{t_2} t^{p'(1/n-1)} dt \right)^{1/p'} \quad (\text{Hardy's inequality}) \\ &\leq c_p \|\nabla f\|_p \left(\int_0^{|t_2-t_1|} t^{p'(1/n-1)} dt \right)^{1/p'} \quad (\text{since } t^{p'(1/n-1)} \text{ decreases}) \\ &= c_{p,n} \|\nabla f\|_p |t_2 - t_1|^{1/n-1/p} \\ &= c_{p,n} \|\nabla f\|_p |t_2 - t_1|^{\alpha/n}. \end{aligned}$$

At this point it is not difficult to reformulate many of the results in this paper using the notion of Lorentz spaces with negative index. As an example we simply state the following result and safely leave the details to the reader.

THEOREM 33. *Let (Ω, d, μ) be a probability metric space that satisfies the relative isoperimetric property and such that*

$$t^{1-1/n} \leq I_{\Omega}(t), \quad t \in (0, 1/2).$$

Then, if $p > n$

$$b_{p,1}^{n/p}(\Omega) \subset L^{\bar{p},1}$$

where $\frac{1}{\bar{p}} = \frac{1}{p} - \frac{1}{n}$. Moreover, if $f \in b_{p,1}^{n/p}(\Omega)$, then $\forall B \subset \Omega$, $f\chi_B \in L^{\bar{p},1}$, and $\|f\chi_B\|_{L^{\bar{p},1}} \leq \|f\|_{b_{p,1}^{n/p}(\Omega)}$, with constants independent of B . In particular, it follows that $f \in C(\Omega)$.

3. An interpolation inequality

In this section we formulate the basic argument of this chapter as in interpolation inequality.

LEMMA 6. *Suppose that (Ω, d, μ) is a probability measure. Let $s < 0, 1 \leq q \leq \infty$, and suppose that $-q' > s$. Then for all $f \in L^1(\Omega)$ we have,*

$$\|f\|_{L^\infty} \leq \left(\frac{-s}{q'}\right)^{1/q'} \|f\|_{L^{[s,q]}} + \|f\|_{L^1}.$$

PROOF. We use the argument of the previous section verbatim. Let $0 < t_1 < t_2 < 1$. By the fundamental theorem of calculus, we have

$$\begin{aligned} |f_\mu^{**}|(t_1) - |f_\mu^{**}|(t_2) &= \int_{t_1}^{t_2} \left(|f_\mu^{**}|(t) - |f_\mu^*|(t)\right) \frac{dt}{t} \\ &= \int_{t_1}^{t_2} \left(|f_\mu^{**}|(t) - |f_\mu^*|(t)\right) t^{1/s} t^{-1/s} \frac{dt}{t} \\ &\leq \left\{ \int_{t_1}^{t_2} \left\{ \left(|f_\mu^{**}|(t) - |f_\mu^*|(t)\right) t^{1/s} \right\}^q \frac{dt}{t} \right\}^{1/q} \left\{ \int_{t_1}^{t_2} t^{-q'/s} \frac{dt}{t} \right\}^{1/q'} \\ &\leq \left(\frac{-s}{q'}\right)^{1/q'} \|f\|_{L^{[s,q]}} |t_1 - t_2|^{-q'/s}. \end{aligned}$$

Therefore letting $t_1 \rightarrow 0^+, t_2 \rightarrow 1^-$, we find

$$\|f\|_{L^\infty} - \|f\|_{L^1} \leq \left(\frac{-s}{q'}\right)^{1/q'} \|f\|_{L^{[s,q]}}.$$

□

4. Further remarks

Good portions of the preceding discussion can be extended to the context of real interpolation spaces. In this framework one can consider spaces that are defined in terms of conditions on $\frac{K(t,f;\bar{X})}{t} - K'(t,f;\bar{X})$, where \bar{X} is a compatible pair of Banach spaces. An example of such construction are the modified Lions-Peetre spaces defined, for example, in [50], [51] and the references therein. The usual conditions defining these spaces are of the form

$$\|f\|_{[X_0, X_1]_{\theta, q}} = \left\{ \int_0^\infty (t^{-\theta} (K(t, f; X_0, X_1) - tK'(t, f; X_0, X_1))^q \frac{dt}{t}) \right\}^{1/q} < \infty,$$

where $\theta \in (0, 1), q \in (0, \infty]$. Adding the end points $\theta = 0, 1$, produces conditions that still make sense and are useful in analysis (cf. [66] and the references therein). Observe that when $\bar{X} = (L^1, L^\infty)$, we have

$$\frac{K(t, f; \bar{X})}{t} - K'(t, f; \bar{X}) = |f|^{**}(t) - |f|^*(t),$$

and therefore

$$[X_0, X_1]_{\theta, q} = L^{[\frac{1}{1-\theta}, q]}.$$

Therefore the discussion in this chapter suggests that it is of interest to consider, more generally, the spaces $[X_0, X_1]_{\theta, q}$, for $\theta \in \mathbb{R}$. In particular this may allow, in some cases, to treat L^p and Lip conditions in a unified manner. For example, in

[29] and [70] results are given that imply that for certain operators T , that include gradients, an inequality of the form

$$\|f\|_{[Y_0, Y_1]_{\theta_0, q_0}} \leq c \|Tf\|_{[X_0, X_1]_{\theta_1, q_1}}$$

can be extrapolated to a family of inequalities that involve the $[Y_0, Y_1]_{\theta_0, q_0}$ spaces defined here. In particular

$$\|f\|_{L^{n'}} \leq c \|\nabla f\|_{L^1}, f \in C_0^1(R^n)$$

implies

$$K(t, f; L^1, L^\infty) - tK'(t, f; L^1, L^\infty) \leq ct^{1/n} K(t, \nabla f; L^1, L^\infty).$$

Thus from one inequality we can extrapolate “all” the classical Sobolev inequalities through the use of the $[Y_0, Y_1]_{\theta, q}$ spaces with θ possibly negative. To pursue this point further would take us too far away from our main concerns in this paper, so we must leave more details and applications for another occasion.

Connection with the work of Garsia and his collaborators

In this section we shall discuss the connection of our results with the work of Garsia and his collaborators (cf. [44], [43], [40], [42], [41], [80], [31]...). We argue that our results can be seen as an extension the work by Garsia [41], [42], and some¹ of the work by Garsia-Rodemich [43], to the metric setting. Indeed, [42], [41] were one of the original motivations behind [64] and some of our earlier writings.

In [42] it is shown that for functions on $[0, 1]$,

$$(1.1) \quad \left. \begin{array}{l} f^*(x) - f^*(1/2) \\ f^*(1/2) - f^*(1-x) \end{array} \right\} \leq \frac{4^{1/p}}{\log \frac{3}{2}} \int_x^1 Q_p(\delta, f) \frac{d\delta}{\delta^{1+1/p}},$$

(see Section 1.1 in Chapter 4 above), and where

$$Q_p(\delta, f) = \left\{ \frac{1}{\delta} \int \int_{|x-y|<\delta} |f(x) - f(y)|^p dx dy \right\}^{1/p}.$$

In particular, if² $p > 1$, and $\int_0^1 Q_p(\delta, f) \frac{d\delta}{\delta^{1+1/p}} < \infty$, then f is essentially continuous, and in fact, *a.e.* $x, y \in [0, 1]$

$$(1.2) \quad |f(x) - f(y)| \leq 2 \frac{4^{1/p}}{\log \frac{3}{2}} \int_0^{|x-y|} Q_p(\delta, f) \frac{d\delta}{\delta^{1+1/p}}.$$

Moreover, in [42] more general moduli of continuity based on Orlicz spaces are considered: for a Young's function A , normalized so that $A(1) = 1$, let

$$Q_A(\delta, f) = \inf \left\{ \lambda > 0 : \frac{1}{\delta} \int \int_{|x-y|<\delta} A \left(\frac{|f(x) - f(y)|}{\lambda} \right) dx dy \leq 1 \right\}.$$

In [42] and Deland [31, (1.1),(1.3)] the following analogues of (1.1) and (1.2) are shown to hold:

$$(1.3) \quad \left. \begin{array}{l} f^*(x) - f^*(1/2) \\ f^*(1/2) - f^*(1-x) \end{array} \right\} \leq \frac{2}{\log \frac{3}{2}} \int_x^1 Q_A(\delta, f) A^{-1} \left(\frac{4}{\delta} \right) \frac{d\delta}{\delta}$$

¹The results of [43], while very similar, are formulated in terms of moduli of continuity that in some cases cannot be readily identified with the ones we consider in this paper.

²The restriction $p > 1$ is necessary to make the result non-trivial since

$$\int_0^1 Q_1(\delta, f) \frac{d\delta}{\delta^2} < \infty,$$

readily implies that f is constant.

and

$$(1.4) \quad |f(x) - f(y)| \leq c \int_0^{|x-y|} Q_A(\delta, f) A^{-1}\left(\frac{4}{\delta}\right) \frac{d\delta}{\delta}.$$

We will show in a moment that our inequalities readily give the following version of (1.3) for all r.i. spaces $X[0, 1]$:

$$(1.5) \quad \left. \begin{array}{l} f^*(x) - f^*(1/2) \\ f^*(1/2) - f^*(1-x) \end{array} \right\} \leq c \int_x^1 \frac{K(\delta, f; X, \dot{W}_X^1)}{\phi_X(\delta)} \frac{d\delta}{\delta}.$$

To relate this inequality to Garsia's results we compare the modulus of continuity to K -functionals. Thus, we let

$$w_A(\delta, f) = \inf \left\{ \lambda > 0 : \sup_{h \leq \delta} \int_0^{1-\delta} A \left(\frac{|f(x+h) - f(x)|}{\lambda} \right) dx \leq 1 \right\}, \quad \delta \in (0, 1).$$

Then, as is well known (cf. [13], [64]),

$$(1.6) \quad K(\delta, f; L_A, \dot{W}_{L_A}^1) \simeq \omega_A(\delta, f),$$

and we have

LEMMA 7. $\sup_{0 < \sigma < \delta} Q_A(\sigma, f) \preceq K(\delta, f; L_A, \dot{W}_{L_A}^1)$.

PROOF. To see this note that, for all $\lambda > 0$, $\delta \in (0, 1)$, we have

$$\begin{aligned} \frac{1}{\delta} \int \int_{\{(x,y) \in [0,1]^2 : |x-y| < \delta\}} A \left(\frac{|f(x) - f(y)|}{2\lambda} \right) dx dy &\leq \frac{1}{\delta} \int_0^\delta \int_0^{1-\delta} A \left(\frac{|f(x+h) - f(x)|}{\lambda} \right) dx dh \\ &\leq \sup_{h \leq \delta} \int_0^{1-\delta} A \left(\frac{|f(x+h) - f(x)|}{\lambda} \right) dx. \end{aligned}$$

Therefore, if we let $\lambda = \omega_A(\delta, f)$, by the definitions,

$$\frac{1}{\delta} \int \int_{\{(x,y) \in [0,1]^2 : |x-y| < \delta\}} A \left(\frac{|f(x) - f(y)|}{2\lambda} \right) dx dy \leq 1,$$

and consequently

$$\begin{aligned} Q_A(\delta, f) &\leq 2\omega_A(\delta, f) \\ &\preceq K(\delta, f; L_A, \dot{W}_{L_A}^1). \end{aligned}$$

□

To complete the picture let us also note that

$$\phi_{L_A}(t) = \frac{1}{A^{-1}(\frac{1}{t})}.$$

We now show in detail (1.5). One technical problem we have to overcome is that the results of this paper do not apply directly for functions on $[0, 1]$, since the isoperimetric profile of $[0, 1]$ is $I(t) \equiv 1$, and therefore I does not satisfy the required hypotheses to apply our general machinery (cf. Condition 1 in Chapter 2, and [69], [70]). Therefore while the inequalities [(1.8), Chapter 1], and their corresponding signed rearrangement variants are valid (cf. Chapter 4), our results cannot be applied directly. However, we will now show that our methods can be readily adapted to yield the one dimensional result as well.

To prove [(1.8), Chapter 1] for $n = 1$, we need to establish the following inequality (compare with [(1.2), Chapter 1, letting formally $I(t) = 1$])

$$f^{**}(t) - f^*(t) \leq t(|f'|)^{**}(t), t \in (0, 1).$$

While [69] formally does not cover this case, it turns out that we can easily prove this inequality directly using the method of “truncation by symmetrization”, which was apparently introduced in [71]. Indeed, a known elementary result of Duff [35] states that

$$\|(f^*)'\|_{L^p[0,1]} \leq \|f'\|_{L^p[0,1]}.$$

The truncation method of [71] (cf. also [36, discussion before Corollaire 2.4]), as it is developed in detail in [60], when applied to the case $p = 1$, yields the corresponding Pólya-Szegő inequality (as formulated in [71])

$$t((f^*)')^{**}(t) \leq t(|f'|)^{**}(t), t \in (0, 1).$$

We can (and will) assume without loss that f is bounded, then (cf. [60]),

$$t((f^*)')^{**}(t) = \int_0^t |(f^*)'| ds = f^*(0) - f^*(t) < \infty.$$

Now, since $f^{**}(0) = f^*(0)$, and f^{**} is decreasing, we have

$$f^{**}(t) - f^*(t) \leq f^{**}(0) - f^*(t) = f^*(0) - f^*(t).$$

Therefore, combining these estimates we arrive at

$$f^{**}(t) - f^*(t) \leq t(|f'|)^{**}(t), t \in (0, 1),$$

as required. Now, the proof Theorem 7 applies without changes to yield

$$f^{**}(t) - f^*(t) \leq c \frac{K(t, f; X, \dot{W}_X^1)}{\phi_X(t)}.$$

Let $t \in (0, \frac{1}{2}]$, using [9, (4.1)] we have

$$f^*(t) - f^*(2t) \leq 2(f^{**}(t) - f^*(t)).$$

On the other hand since $K(t, f; X, \dot{W}_X^1)$ increases, and $\frac{t}{\phi_X(t)}$ increases, we get

$$\begin{aligned} \frac{K(t, f; X, \dot{W}_X^1)}{\phi_X(t)} &= \frac{1}{t} K(t, f; X, \dot{W}_X^1) \frac{t}{\phi_X(t)} \\ &\leq \frac{1}{t \ln 2} \int_t^{2t} K(t, f; X, \dot{W}_X^1) \frac{s}{\phi_X(s)} \frac{ds}{s} \\ &= \frac{2}{2t \ln 2} \int_t^{2t} K(t, f; X, \dot{W}_X^1) \frac{s}{\phi_X(s)} \frac{ds}{s} \\ &\leq \frac{2}{\ln 2} \int_t^{2t} \frac{K(s, f; X, \dot{W}_X^1)}{\phi_X(s)} \frac{ds}{s}. \end{aligned}$$

Combining inequalities we see that

$$f^*(t) - f^*(2t) \leq \frac{4}{\ln 2} \int_t^{2t} \frac{K(s, f; X, \dot{W}_X^1)}{\phi_X(s)} \frac{ds}{s}, t \in (0, \frac{1}{2}].$$

Letting $t = 2^n t$ and summing up to $N(t)$ such that $2^{N+1}t \in [\frac{1}{2}, 1]$, we get

$$\begin{aligned} f^*(t) - f^*\left(\frac{1}{2}\right) &\leq f^*(t) - f^*(2^{N+1}t) \\ &\leq c \int_t^1 \frac{K(s, f; X, \dot{W}_X^1)}{\phi_X(s)} \frac{ds}{s}, \quad t \in (0, \frac{1}{2}). \end{aligned}$$

When the inequality is applied to $-f$ we obtain the second half of Garsia's inequality.

In particular, our results thus give versions of (1.1), (1.2), (1.3), but replacing $Q_A(\delta, f)$ with the usual modulus of continuity $K(\delta, f; L_A, \dot{W}_{L_A}^1)$. We also note that Deland [31] found the following improvement to (1.3)

$$\left. \begin{aligned} f^*(x) - f^*(1/2) \\ f^*(1/2) - f^*(1-x) \end{aligned} \right\} \preceq \int_x^1 Q_A(\delta, f) dA^{-1}\left(\frac{c}{\delta}\right), \quad 0 < x < 1/2.$$

This is of particular interest when dealing with the space $X = e^{L^2}$. Indeed, in this case $A(t) = e^{t^2} - 1$, and therefore

$$\phi_X(t) = \frac{1}{(\ln \frac{e}{t})^{1/2}}.$$

Consequently, from (1.4) (or (1.5)) one finds that a sufficient condition for continuity can be formulated as: There exists $0 < a < 1, c > 0$, such that

$$(1.7) \quad \int_0^a Q_A(\delta, f) \left(\ln \frac{c}{\delta}\right)^{1/2} \frac{d\delta}{\delta} < \infty.$$

On the other hand, Deland's improved condition for continuity replaces (1.7) by

$$(1.8) \quad \int_0^a Q_A(\delta, f) \frac{d\delta}{(\ln \frac{c}{\delta})^{1/2} \delta} < \infty.$$

In our formulation (1.8) corresponds to a condition of the form

$$\int_0^a K(\delta, f; L_A, \dot{W}_{L_A}^1) d\left(\frac{1}{\phi_A(t)}\right) < \infty.$$

While we don't have any new insight to add to Deland's improvement we should add here that Deland's improvement is automatic for spaces far away from L^∞ , in the sense that $\underline{\alpha}_{\Lambda(X)} > 0$. Indeed, we have

LEMMA 8. *Suppose that $X = X[0, 1]$ is a r.i. space such that $\underline{\alpha}_{\Lambda(X)} > 0$. Then there exists a re-norming of X , that we shall call \bar{X} , such that*

$$(1.9) \quad \int_0^1 K(\delta, f; \bar{X}, \dot{W}_{\bar{X}}^1) d\left(\frac{1}{\phi_{\bar{X}}(\delta)}\right) < \infty \iff \int_0^1 \frac{K(\delta, f; X, \dot{W}_X^1)}{\phi_X(\delta)} \frac{d\delta}{\delta} < \infty.$$

PROOF. Let $\bar{\phi}(t) = \int_0^t \phi_X(s) \frac{ds}{s}$, then, since $\frac{\phi_X(s)}{s}$ decreases, we have $\bar{\phi}(t) \geq \phi_X(t)$, and

$$([-\bar{\phi}(t)]^{-1})' = \frac{1}{\bar{\phi}(t)^2} \frac{\phi_X(t)}{t} \leq \frac{1}{\bar{\phi}(t)t} \leq \frac{1}{t\phi_X(t)}.$$

Moreover, since $\underline{\alpha}_{\Lambda(X)} > 0$, we have (cf. [85, Lemma 2.1])

$$\bar{\phi}(t) \preceq \phi_X(t).$$

Therefore there exists an equivalent re-norming of X , which we shall call \bar{X} , such that

$$\phi_X(t) \simeq \phi_{\bar{X}}(t) = \bar{\phi}(t).$$

Moreover, we clearly have

$$K(\delta, f; X, \dot{W}_X^1) \simeq K(\delta, f; \bar{X}, \dot{W}_{\bar{X}}^1).$$

We can also see that,

$$\begin{aligned} \left([\phi_{\bar{X}}(t)]^{-1} \right)' &= ([-\bar{\phi}(t)]^{-1})' \\ &= \bar{\phi}(t)^{-2} \frac{\phi_X(t)}{t} \\ &\simeq \frac{1}{(\phi_X(t))^2} \frac{\phi_X(t)}{t} \\ &\simeq \frac{1}{\phi_X(t)t}. \end{aligned}$$

Consequently (1.9) holds when $\underline{\alpha}_{\Lambda(X)} > 0$. \square

On the other hand Deland's improvement does not follow from the previous Lemma, since from the point of view of the theory of indices $\underline{\alpha}_{\Lambda(e^{L^2})} = 0$. For more details on how to overcome this difficulty for spaces close to L^∞ we must refer to Deland's thesis [31].

For applications to Fourier series, the appropriate moduli of continuity defined for periodic functions on, say, $[0, 2\pi]$, are defined by (cf. [42], [31, (1.1), (1.3)])

$$W_A(h, f) = \inf \left\{ \lambda > 0 : \int_0^{2\pi} A \left(\frac{|f(x+h) - f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$

Then, we also have

$$\begin{aligned} \sup_{\sigma < \delta} Q_A(\sigma, f) &\preceq K(\delta, f; L_A[0, 2\pi], \dot{W}_{L_A}^1[0, 2\pi]) \\ &\simeq \sup_{h \leq \delta} W_A(h, f). \end{aligned}$$

It follows from our work that the results of [42] can be now extended to r.i. spaces. In this connection we note that (just like in [42] for L^p spaces) one could also use the boundedness of the Hilbert transform on r.i. spaces where one has control of the Boyd indices (cf. [21], [13]). However, to continue with this topic will take us too far away from our main concerns here so we must leave the discussion for another occasion.

For further applications to: the path continuity of stochastic processes, Fourier series, random Fourier series and embeddings we refer to [43], [42], [41], [31] and the references therein. Moreover, under suitable assumptions on the connection between the isoperimetric profile and the measure of balls (cf. [91]) one can also formulate the Besov conditions as entropy conditions as it is customarily done in probability (cf. the discussion in Pisier [83, cf. Remarque, p 14.])

Appendix: Some remarks on the calculation of K -functionals

1. Introduction

It seemed to us useful to collect for our reader some known computations of K -functionals of the form $K(t, f; X(\Omega), \dot{W}_X^1(\Omega))$, where X is a r.i. space. We don't claim any originality, but we provide detailed proofs when we could not find suitable references.

In the Euclidean case, for smooth (Lip) domains, these estimates are well known for L^p spaces (cf. [13], [54], [93]), and can be readily extended to r.i. spaces (cf. [64]):

$$\begin{aligned} K(t, f; X(\Omega), \dot{W}_X^1(\Omega)) &\simeq \omega_X(t, f) \\ &= \sup_{|h| \leq t} \|(f(\cdot + h) - f(\cdot)) \chi_{\Omega(h)}\|_X \end{aligned}$$

where

$$\Omega(h) = \{x \in \Omega : x + th \in \Omega, 0 \leq t \leq 1\}.$$

Consider $(\mathbb{R}^n, |\cdot|, d\gamma_n)$, i.e. \mathbb{R}^n with Gaussian measure. The fact that this measure is not translation invariant makes the computation of the K -functional somewhat more complicated. We discuss the necessary modifications in some detail for $n = 1$.

We consider spaces on $(\mathbb{R}, |\cdot|, d\gamma_1)$. Let $p \in [1, \infty]$, and let

$$K_\gamma(t, f, L^p, \dot{W}_p^1) = \inf \left\{ \|f - g\|_{L^p(\mathbb{R}, d\gamma_1)} + t \|g'\|_{L^p(d\gamma_1)} \right\}.$$

This functional was studied by the approximation theory community (cf. Ditzian-Totik [32], Ditzian-Lubinsky [34] and the references therein). For example, from [32, page 183], we have

$$\begin{aligned} K_\gamma(t, f, L^p, \dot{W}_p^1) &\simeq \sup_{0 < h \leq t} \|f(\cdot + h) - f(\cdot)\|_{L^p((-\frac{1}{2h}, \frac{1}{2h}), d\gamma_1)} + \inf_c \|f - c\|_{L^p((\frac{1}{2t}, \infty), d\gamma_1)} \\ (1.1) \quad &+ \inf_c \|f - c\|_{L^p((-\infty, -\frac{1}{2t}), d\gamma_1)} \end{aligned}$$

The main part of the right hand side of (1.1) is the modulus

$$\Omega_\gamma(t, f) = \sup_{0 < h \leq t} \|f(\cdot + h) - f(\cdot)\|_{L^p((-\frac{1}{2h}, \frac{1}{2h}), d\gamma_1)}.$$

Indeed, $\Omega_\gamma(t, f)$ controls the characterization of the corresponding interpolation spaces. For example, it follows (cf. [32, Theorem 11.2.5]) that for $\theta \in (0, 1)$,

$$(1.2) \quad K_\gamma(t, f, L^p, \dot{W}_p^1) = O(t^\theta) \iff \Omega_\gamma(t, f) = O(t^\theta).$$

More generally, a similar result holds for $(\mathbb{R}, d\gamma_\lambda)$, where for $\lambda > 1$, $d\gamma_\lambda(x) = e^{-x^\lambda} dx$. Indeed, in this case (1.1) holds replacing $\frac{1}{2h}$ throughout by $\frac{1}{\lambda h^{1/(1-\lambda)}}$:

$$(1.3) \quad K_{\gamma_\lambda}(t, f, L^p, \dot{W}_p^1) \simeq \sup_{0 < h \leq t} \|f(\cdot + h) - f(\cdot)\|_{L^p\left(-\frac{1}{\lambda h^{1/(1-\lambda)}}, \frac{1}{\lambda h^{1/(1-\lambda)}}\right), d\gamma_\lambda} + \\ \inf_c \|f - c\|_{L^p\left(\frac{1}{\lambda t^{1/(1-\lambda)}}, \infty\right), d\gamma_\lambda} + \inf_c \|f - c\|_{L^p\left(-\infty, \frac{1}{\lambda t^{1/(1-\lambda)}}\right), d\gamma_\lambda}.$$

Again the main part of the right hand side is the modulus of continuity

$$\Omega_{\gamma_\lambda}(t, f) = \sup_{0 < h \leq t} \|f(\cdot + h) - f(\cdot)\|_{L^p\left(-\frac{1}{\lambda h^{1/(1-\lambda)}}, \frac{1}{\lambda h^{1/(1-\lambda)}}\right), d\gamma_\lambda}.$$

Likewise the analogue of (1.2) holds.

More generally, the estimates above have been extended to the class of the so called “Freud weights” of the form $w(x) = e^{Q(x)}$. Here we assume that Q is a given function in $C^1(\mathbb{R})$ such that Q is even, $\lim_{x \rightarrow \infty} Q'(x) = \infty$, and such that there exists $A > 0$, such that $Q'(x+1) \leq AQ'(x)$, for all $x > 0$. For complete details we refer again to [32].

Although one would expect that the n -dimensional extensions of the K -functional estimates above should not be very difficult, we have not been able find references, even after consultation with many experts. On the other hand, as is well known, one can avoid this difficulty through the use of an alternate characterization of K -functionals for Gaussian measure using appropriate semigroups. We provide some details in the next sections.

For the last section of this chapter, connecting semigroups and Gaussian Besov spaces, we are grateful to Stefan Geiss and Alessandra Lunardi for precious information. In particular, for pointing out the relevant literature. In this last regard we also refer to the recent paper by Geiss-Toivola [46]. In connection with this last section we should mention the recent formulation of fractional Poincaré inequalities in [79].

2. Semigroups and Interpolation

A family $\{G(t)\}_{t>0}$ of operators on a Banach space A is called an equibounded, strongly continuous semigroup if the following conditions are satisfied:

- (i) $G(t+s) = G(t)G(s)$
- (ii) There exists $M > 0$ such that $\sup_{t>0} \|G(t)\|_{A \rightarrow A} \leq M$
- (iii) $\lim_{t \rightarrow 0} \|G(t)a - a\|_A = 0$, for $a \in A$.

The infinitesimal generator Λ is defined on

$$D(\Lambda) = \{a \in A : \lim_{t \rightarrow 0} \frac{G(t)a - a}{t} \text{ exists}\}$$

by

$$\Lambda a = \lim_{t \rightarrow 0} \frac{G(t)a - a}{t}.$$

We consider

$$K(t, a; A, D(\Lambda)) = \inf\{\|a_0\|_A + t\|\Lambda a_1\|_A : a = a_0 + a_1\}.$$

For equibounded strongly continuous semigroups we have the well known estimate, apparently going back to Peetre [82] (cf. [14], [33], [81])

$$(2.1) \quad K(t, a; A, D(\Lambda)) \simeq \sup_{0 < s \leq t} \|(G(s) - I)a\|_A,$$

where I = identity operator on A . The proof can be accomplished using the decomposition

$$a = \underbrace{\left(a - \frac{1}{t} \int_0^t G(s)ads\right)}_{a_0 \in A} + \underbrace{\frac{1}{t} \int_0^t G(s)ads}_{a_1 \in D(\Lambda)}.$$

Note that the right hand side of (2.1) should thought as a generalized modulus of continuity which in the classical case corresponds to the semigroup of translations $G(s)f = f(s + \cdot)$.

In [33, Corollary 7.2] the following alternate estimates were pointed out

$$\begin{aligned} K(t, a; A, D(\Lambda)) &\simeq \frac{1}{t} \int_0^t \|(G(s) - I)a\|_A ds \\ &\simeq \frac{1}{t} \left\| \int_0^t (G(s) - I)ads \right\|_A \\ &\simeq \frac{1}{t} \left\| \int_{t/2}^t (G(s) - I)ads \right\|_A \\ &\simeq \frac{1}{t} \int_{t/2}^t \|(G(s) - I)ads\|_A. \end{aligned}$$

The preceding estimates can be further improved under more restrictions on the semigroups. Recall that a semigroup is said to be holomorphic if:

$$(i) \quad G(t)a \in D(\Lambda) \quad \text{for all } a \in A,$$

and

$$(ii) \quad \text{There exists a constant } C > 0 \text{ such that } \|\Lambda G(t)a\|_A \leq C \frac{\|a\|_A}{t}, \forall a \in A, t > 0.$$

In [33] it is shown that for holomorphic semigroups we have the following improvement of (2.1)

$$(2.2) \quad K(t, a; A, D(\Lambda)) \approx \|(G(t) - I)a\|_A.$$

Peetre [81, page 33] pointed out, without proof, that for holomorphic semigroups we also have

$$K(t, a; A, D(\Lambda)) \simeq \sup_{s \leq t} \|\Lambda G(s)a\|_A.$$

However, we can only prove a somewhat weaker result here.

LEMMA 9. *Suppose that $\{G(t)\}_{t>0}$ is an holomorphic semigroup on a Banach space A . Let $c_1 > 1$, be such that for all $t > 0$, and for all $a \in A$ (cf. (2.2) above),*

$$\frac{1}{c_1} \|(G(t) - I)a\|_A \leq K(t, a; A, D(\Lambda)) \leq c_1 \|(G(t) - I)a\|_A.$$

Then, there exist absolute constants $c_2(m), c_3(m)$ such that for all $t > 0$, for all $a \in A$, for all $m \geq 2$,

$$(2.3) \quad K(t, a; A, D(\Lambda)) - c_1^2 K\left(\frac{t}{m}, a; A, D(\Lambda)\right) \leq c_2(m) \sup_{s \leq t} \|\Lambda G(s)a\|_A \leq c_3(m) K(t, a; A, D(\Lambda)).$$

PROOF. It is easy to show that there exists an absolute constant $C > 0$ such that

$$(2.4) \quad \sup_{s \leq t} \|\Lambda G(s)a\|_A \leq CK(t, a; A, D(\Lambda)).$$

Indeed, let $a = a_0 + a_1$, be any decomposition with $a_0 \in A$, $a_1 \in D(\Lambda)$. Then, using the properties of holomorphic semigroups, we have

$$\begin{aligned} \sup_{s \leq t} s \|\Lambda G(s)a\|_A &\leq \sup_{s \leq t} s \|G(s)\Lambda a_0\|_A + \sup_{s \leq t} s \|G(s)\Lambda a_1\|_A \\ &\leq C(\|a_0\|_A + t \|\Lambda a_1\|_A). \end{aligned}$$

Consequently, (2.4) follows by taking infimum over all such decompositions.

We now prove the left hand side of (2.3). Observe that, for $t > 0$ we have $G(t)a \in D(\Lambda)$, therefore we can write $\frac{d}{dt}(G(t)a) = \Lambda G(t)a$. Consequently, for all $m \geq 2$,

$$\begin{aligned} K(t, a; A, D(\Lambda)) &\leq c_1 \|(G(t) - I)a\|_A \\ &\leq c_1 \left\| \int_0^{t/m} \Lambda G(s)ads \right\|_A + c_1 \left\| \int_{t/m}^t \Lambda G(s)ads \right\|_A \\ &= c_1 \|(G(t/m) - I)a\|_A + c_1 \left\| \int_{t/2}^t \frac{s}{s} \Lambda G(s)ads \right\|_A \\ &\leq c_1 \|(G(t/m) - I)a\|_A + c_1 \int_{t/m}^t \frac{1}{s} \|s \Lambda G(s)a\|_A ds \\ &\leq c_1 \|(G(t/m) - I)a\|_A + c_1 \frac{m}{t} \sup_{s \leq t} \|s \Lambda G(s)a\|_A \frac{(m-1)t}{m} \\ &\leq c_1^2 K\left(\frac{t}{m}, a; A, D(\Lambda)\right) + c_1(m-1) \sup_{s \leq t} \|s \Lambda G(s)a\|_A, \end{aligned}$$

as we wished to show. \square

Recall the definition of real interpolation spaces. Let $\theta \in (0, 1)$, $q \in (0, \infty)$,

$$(A, D(\Lambda))_{\theta, q} = \left\{ a \in A : \|a\|_{(A, D(\Lambda))_{\theta, q}}^q = \int_0^\infty (t^{-\theta} K(s, a; A, D(\Lambda)))^q \frac{dt}{t} < \infty \right\},$$

and

$$(A, D(\Lambda))_{\theta, \infty} = \left\{ a \in A : \|a\|_{(A, D(\Lambda))_{\theta, \infty}} = \sup_{t > 0} \{t^{-\theta} K(s, a; A, D(\Lambda))\} < \infty \right\}.$$

From the previous Lemma we see that

PROPOSITION 5. Suppose that $\{G(t)\}_{t>0}$ is an holomorphic semigroup on a Banach space A . Then $(A, D(\Lambda))_{\theta, q}$ can be equivalently described by

$$(A, D(\Lambda))_{\theta, q} = \left\{ a : \left\{ \int_0^\infty \left(t^{-\theta} \sup_{s \leq t} \|\Lambda G(s)a\|_A \right)^q \frac{dt}{t} \right\}^{1/q} < \infty \right\},$$

with the obvious modification if $q = \infty$, and where the constants of the underlying norm equivalences depend only on θ .

PROOF. One part follows readily from (2.4). For the less trivial inclusion we proceed as follows. Given $\theta \in (0, 1)$, select m such that $m^{-\theta}c_1^2 < 1$. Then from Lemma 9, there exists an absolute $c_2(m) > 0$ such that

$$K(t, a; A, D(\Lambda)) \leq c_2(m) \sup_{s \leq t} \|\Lambda G(s)a\|_A + c_1^2 K\left(\frac{t}{m}, a; A, D(\Lambda)\right)$$

Thus

$$\begin{aligned} \left(\int_0^\infty (t^{-\theta} K(t, a; A, D(\Lambda)))^q \frac{dt}{t} \right)^{1/q} &\leq c_2(m) \left(\int_0^\infty \left(t^{-\theta} \sup_{s \leq t} \|\Lambda G(s)a\|_A \right)^q \frac{dt}{t} \right)^{1/q} \\ &\quad + c_1^2 \left(\int_0^\infty (t^{-\theta} K\left(\frac{t}{m}, a; A, D(\Lambda)\right))^q \frac{dt}{t} \right)^{1/q} \\ &= c_2(m) \left(\int_0^\infty \left(t^{-\theta} \sup_{s \leq t} \|\Lambda G(s)a\|_A \right)^q \frac{dt}{t} \right)^{1/q} \\ &\quad + c_1^2 m^{-\theta} \left(\int_0^\infty (t^{-\theta} K(t, a; A, D(\Lambda)))^q \frac{dt}{t} \right)^{1/q}. \end{aligned}$$

Hence,

$$\left(\int_0^\infty (t^{-\theta} K(t, a; A, D(\Lambda)))^q \frac{dt}{t} \right)^{1/q} \leq (1 - c_1^2 m^{-\theta})^{-1} c_2(m) \left(\int_0^\infty \left(t^{-\theta} \sup_{s \leq t} \|\Lambda G(s)a\|_A \right)^q \frac{dt}{t} \right)^{1/q}.$$

□

We have the following well known result (cf. [22], [14])

THEOREM 34. *Let $\{G(t)\}_{t \geq 0}$ be an equibounded, strongly continuous semigroup on the Banach space A . Let $\theta \in (0, 1)$, $q \in (0, \infty]$; then (with the usual modifications when $q = \infty$)*

(i)

$$(A, D(\Lambda))_{\theta, q} = \left\{ a : \left\{ \int_0^\infty \left(t^{-\theta} \sup_{0 < s \leq t} \|G(s)a - a\|_A \right)^q \frac{dt}{t} \right\}^{1/q} < \infty \right\}.$$

(ii) *Moreover, if the semigroup is analytic then we also have the following characterizations (with the usual modifications when $q = \infty$)*

(ii₁)

$$(A, D(\Lambda))_{\theta, q} = \left\{ a : \left\{ \int_0^\infty (t^{-\theta} \|G(t)a - a\|_A)^q \frac{dt}{t} \right\}^{1/q} < \infty \right\},$$

(ii₂)

$$(A, D(\Lambda))_{\theta, q} = \left\{ a \in A : \int_0^\infty \left(t^{-\theta} \sup_{s \leq t} s \|\Lambda G(s)a\|_A \right)^q \frac{dt}{t} < \infty \right\},$$

(ii₃)

$$(A, D(\Lambda))_{\theta, q} = \left\{ a \in A : \int_0^\infty (t^{1-\theta} \|\Lambda G(t)a\|_A)^q \frac{dt}{t} < \infty \right\}.$$

PROOF. The characterizations (i)–(ii₁)–(ii₂) follow (respectively) from (2.1), (2.2) and Proposition 5. To prove (ii₃) we remark that, on the one hand,

$$\int_0^\infty (t^{1-\theta} \|\Lambda G(t)a\|_A)^q \frac{dt}{t} \leq \int_0^\infty \left(t^{-\theta} \sup_{s \leq t} \| \Lambda G(s)a \|_A \right)^q \frac{dt}{t}.$$

On the other hand, since $\frac{d}{dt}(G(t)a) = \Lambda G(t)a$,

$$\begin{aligned} \int_0^\infty (t^{-\theta} \|G(t)a - a\|_A)^q \frac{dt}{t} &= \int_0^\infty \left(t^{-\theta} \left\| \int_0^t \Lambda G(s)a ds \right\|_A \right)^q \frac{dt}{t} \\ &\leq \int_0^\infty \left(t^{-\theta} \int_0^t \|\Lambda G(s)a\|_A ds \right)^q \frac{dt}{t} \\ &\leq c_{\theta,q} \int_0^\infty (t^{1-\theta} \|\Lambda G(t)a\|_A)^q \frac{dt}{t}, \end{aligned}$$

where the last step follows from Hardy's inequality. \square

REMARK 18. *Related interpolation spaces (obtained by the “complex method”) can be characterized, under suitable conditions, using functional calculus. By the known relations between these different interpolation methods one can obtain further characterizations and embedding theorems for the real method (cf. [95]). In this setting fractional powers of the infinitesimal generator Λ , play the role of fractional derivatives. We must refer to [93] and [81] for a complete treatment.*

3. Specific Semigroups

Two basic examples of semigroups on $L^p(\mathbb{R}^n, d\gamma_n)$, which are relevant for this paper are given by

1. Ornstein-Uhlenbeck semigroup, defined by

$$G(t)f(x) = (1 - e^{-2t})^{-n/2} \int e^{-\frac{e^{-2t}(|x|^2 + |y|^2 - 2\langle x, y \rangle)}{1 - e^{-2t}}} f(y) d\gamma_n(y),$$

with generator

$$\Lambda = \frac{1}{2} \Delta_x - \langle x, \nabla_x \rangle.$$

2. Poisson-Hermite semigroup

$$P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} G\left(\frac{t^2}{4s}\right) f(x) ds,$$

with generator

$$\Lambda_{1/2} = -(-\Lambda)^{1/2}.$$

For example, P_t on $L^\infty(\mathbb{R}^n)$ is analytic although not strongly continuous. Restricting P_t to $\widetilde{L^\infty(\mathbb{R}^n)}$, the subspace of elements of $L^\infty(\mathbb{R}^n)$ such that $\lim \|P_t f - f\|_\infty = 0$, remedies this deficiency and we have (cf. [93])

$$(L^\infty(\mathbb{R}^n), d\gamma_n, D(\Lambda_{1/2}))_{\theta, \infty} = (\widetilde{L^\infty(\mathbb{R}^n)}, d\gamma_n, D(\Lambda_{1/2}))_{\theta, \infty} = Lip_\theta(\mathbb{R}^n).$$

In particular, it follows from Theorem 34 that $f \in Lip_\theta(\mathbb{R}^n)$, iff

$$\|P_t f - f\|_\infty = O(t^\theta).$$

For other characterizations of Besov spaces we must refer to [81], [93] and the references therein. For a treatment of fractional derivatives in Gaussian Lipschitz spaces using semigroups and classical analysis we refer to [45].

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